8 LQR with average-cost optimality

Since last week, we have been considering an LQR problem with average-cost optimality described by the following components:
System's dynamics:

$$X_{t+1} = AX_t + BU_t + W_t \tag{1}$$

, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, X_t \in \mathbb{R}^n, U_t \in \mathbb{R}^m$, and $W_t \in \mathbb{R}^n$. $\{W_t\}_{t\geq 0}$ is independent and identically distributed with $\mathbf{E}[W_t] = 0$ and $\mathrm{Cov}(W_t) = \Sigma(\succeq 0)$ at each time.
Cost per time step:

$$c(x,u) = x^T Qx + u^T Ru \tag{2}$$

, where $Q = Q^T \succeq 0$ and $R = R^T \succeq 0$.
Average-cost optimality:
The optimal policy and cost are found by minimizing

$$\bar{J}(x;g) = \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}^g \left[\sum_{t=0}^{T-1} c(X_t,U_t)|X_0 = x\right] \tag{3}$$

over a "suitable class of polices". This point will be further discussed later.
Average Cost Optimality Equation (ACOE)
Now, we discuss the problem using a canonical triple (λ, h, ϕ). From the discussion last week, the elements of the canonical triple, $\lambda > 0, h : \mathcal{X} \to \mathbb{R}$, $\phi : \mathcal{X} \to \mathcal{U}$, and stationary policy $\phi^\infty = (\phi, \phi, \phi, \ldots)$ have the following relations:

$$J_T(x,h;\phi^\infty) = \mathbf{E}^{\phi^\infty}\left[\sum_{t=0}^{T-1} c(X_t,U_t) + h(X_T)|X_0 = x\right] \tag{4}$$

$$= V_T(x,h) \quad \left(:= \inf_g J_T(x,h;g)\right) \tag{5}$$

$$= T\lambda + h(x) \tag{6}$$

The pair (λ, h) can be found by the average cost optimality equation (ACOE)

$$\lambda + h(x) = \min_{u\in\mathbb{R}^m}\{c(x,u) + \mathbf{E}h(Ax + Bu + w)\} \tag{7}$$

To solve the ACOE, we first assume that the function $h(x)$ has a quadratic form of x, or

$$h(x) = x^T Kx \tag{8}$$

for some $K = K^T \succeq 0$ (Recall that the cost function was quadratic for LQR problems with finite horizon). Then, the ACOE can be rewritten as

$$\lambda + x^T Kx = \min_{u\in\mathbb{R}^m}\{x^T Qx + u^T Ru + (Ax + Bu)^T K(Ax + Bu)\} + \mathrm{tr}(K\Sigma) \tag{9}$$

By defining $F(x,u)$ as

$$F(x,u) = x^T Qx + u^T Ru + (Ax+Bu)^T K(Ax+Bu) \tag{10}$$
$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q + A^T KA & A^T KB \\ B^T KA & R + B^T KB \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{11}$$

and a matrix M as

$$M = \begin{bmatrix} Q + A^T KA & A^T KB \\ B^T KA & R + B^T KB \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{12}$$

, where $M_{11} = Q + A^T KA$, $M_{12} = A^T KB$, $M_{21} = B^T KA$, and $M_{22} = R + B^T KB \succeq 0$. Because $M_{22} \succeq 0$, the Schur complement

$$S = M_{11} - M_{12} M_{22}^{-1} \tag{13}$$
$$= Q + A^T KA - A^T KB(R + B^T KB)^{-1} B^T KA \tag{14}$$

is also positive semi-definnite ($S \succeq 0$), and

$$\min_{u \in \mathbb{R}^m} F(x,u) = x^T Sx \tag{15}$$
$$u^* = -(R + B^T KB)^{-1} B^T KAx = Gx \tag{16}$$

, where $G := -(R + B^T KB)^{-1} B^T KA$.
Using the Schur complement, the ACOE can be rewritten as

$$\lambda + x^T Kx = \mathrm{tr}(K\Sigma) + x^T Sx \quad \text{for any } x \in \mathbb{R}^n \tag{17}$$

Assuming the K we seek exists, $\lambda = \mathrm{tr}(K\Sigma)$ and $K = S$. Therefore the positive semi-definite symmetric matrix K has to solve the following equation known as discrete algebraic Riccati equation (DARE):

$$K = A^T[K - KB(R + B^T KB)^{-1} B^T K]A + Q \tag{18}$$

Then,

$$\phi(x) = G(x) = -(R + B^T KB)^{-1} B^T KAx \tag{19}$$

is a candidate optimal policy, satisfying

1. For a quadratic function $h(x) = x^T Kx$ with symmetric positive semi-definite K,

$$\lim_{T \to \infty} \frac{1}{T} \mathbf{E}^{\phi}[h(X_T) | X_0 = x] = 0 \tag{20}$$

2. $\phi^{\infty} = (\phi, \phi, \ldots)$ will be AC optimal among all g such that

$$\frac{1}{T} \mathbf{E}^{g}[h(X_T) | X_0 = x] \to 0 \tag{21}$$

Assumption 8.1 *Two assumptions are made in the following discussion:*

1. *(A, B) is controllable, i.e.*

$$\text{rank}\begin{bmatrix} A & AB & \dots & A^{n-1}B \end{bmatrix} = n \tag{22}$$

2. *There exist C such that $Q = C^T C$ and (A, C) is observable, i.e.*

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{23}$$

Theorem 8.1 *Under assumption 8.1,*

1. *The DARE has a unique solution $K(= K^T \succeq 0)$.*

2. *The matrix $D = A + BG$ is stable, where G was defined in eq.(16).*

The stability of the matrix D implies that all eigenvalues of D lie inside the unit circle in the complex plane ($|\lambda_i| < 1$ for $i \in \{1, 2, \dots, n\}$). Such matrix D goes to zero matrix by multiplying infinitely many times ($D^k \to 0$ as $k \to 0$). Also note that the matrix D is a closed-loop matrix obtained by substituting $u = \phi(x) = Gx$ to the sustem's equation in eq.(1). The stability also implies the following claim:
Claim: Under a policy ϕ,

$$\sup_{t \geq 0} \mathbf{E} \left\| X_t^2 \right\| < \infty \Rightarrow \frac{1}{T}\mathbf{E}[h(X_T)|X_0 = x] \xrightarrow{T \to \infty} 0 \tag{24}$$

This claim is explained using the maximum eignvalue of K, λ_{max}, and the upperbound of $\left\| X_t^2 \right\|$, M, as follows.

$$h(x) = x^T K x \leq \lambda_{max} M < \infty \tag{25}$$

Because $h(x)$ is upperbounded, $\frac{1}{T}\mathbf{E}[h(X_T)|X_0 = x]$ goes to 0 as T goes to infinity.

Now we prove Theorem 8.1.

Proof: Consider the Riccati recursion.

- Initialize K by an arbitrary symmetric positive semi-definite matrix such that $K_0 = K_0^T \succeq 0$.

- Update K recursively by the following equation:

$$K_{t+1} = A^T[K_t - K_t B(R + B^T K_t B)^{-1} B^T K_t]A + Q \tag{26}$$

Also, the K matrix after t-th Riccati recursion is denoted as $K_t(K_0)$ to keep track of the initial K matrix.

Using the Riccati recursion Theorem 8.1 is proved by the following five steps.
Step 1: Consider a case when $K_0 = 0$, and prove that $K_t(0)$ converges to a symmetric positive semi-definite matrix K as t goes to infinity.
Step 2: Prove that $D = A + BG$ is stable.
Step 3: Prove that $K \succ 0$.
Step 4: $K_t(K_0) \to K$ as $t \to \infty$ for any $K_0 = K_0^T \geq 0$.
Step 5: K is unique.

Proof of Step 1: Consider a deterministic control problem of minimizing a cost function

$$\sum_{s=0}^{t-1}[x_s^T Q x_s + u_s^T R u_s] \tag{27}$$

such that the initial state is x_0 (fixed) and

$$x_{s+1} = Ax_s + Bu_s \quad (s = 0, 1, \ldots, t-1) \tag{28}$$

Claim: The optimal cost of the deterministic problem can be expressed as

$$x_0^T K_t(0) x_0 \quad \text{for any } t \tag{29}$$

and satisfies
a) $x_0^T K_{t+1}(0) x_0 \geq x_0^T K_t(0) x_0$ (nondecreasing), and
b) $\sup_{t\geq 0} x_0^T K_t(0) x_0 < \infty$
Claim a) is proved by fixing any control sequence $u_0, u_1, \ldots, u_t \in \mathbb{R}^m$ and applying eq.(27):

$$\begin{aligned} x_0^T K_{t+1}(0) x_0 &= \sum_{s=0}^{t}[x_s^T Q x_s + u_s^T R u_s] &(30)\\ &\geq \sum_{s=0}^{t-1}[x_s^T Q x_s + u_s^T R u_s] &(31)\\ &= x_0^T K_t(0) x_0 &(32)\end{aligned}$$

To prove **Claim b)**, the assumption of controllability is used. By controllability, there exist a control sequence $v_0, v_1, \ldots, v_{n-1} \in \mathbb{R}^m$ such that

$$x_{s+1} = Ax_s + Bv_s \quad (s = 0, 1, \ldots, n-1) \tag{33}$$

gives $x_n = 0$. The cost of the control

$$\bar{u}_t = \begin{cases} v_t & t = 0, 1, \ldots, n-1 \\ 0 & t \geq n \end{cases} \tag{34}$$

is upperbounded by

$$x_0^T K_T(0) x_0 \leq \sum_{s=0}^{n-1}\{x_s^T A x_s + u_s^T R u_s\} < \infty \tag{35}$$

Using the **Claims a)-b)**, $\{x_0^T K_t(0) x_0\}_{t\geq 0}$ is a nondecreasing sequence bounded from above, and therefore has a limit. Therefore, there exist a matrix $K \in \mathbb{R}^{m\times n}$ such that

$$\lim_{t\to\infty} x_0^T K_t(0) x_0 = x_0^T K x_0 \tag{36}$$

Considering vectors e_i ($i \in \{1, 2, \ldots, n\}$) such that the i-th element of e_i is 1 and all the other elements are 0. By taking the initial condition x_0 as e_i,

$$\lim_{t\to\infty} e_i^T K_t(0) e_i = \lim_{t\to\infty} [TK_t(0)]_{ii} \tag{37}$$

$$= e_i^T K e_i = K_{ii} \tag{38}$$

By setting $x_0 = e_i + e_j$,

$$\lim_{t\to\infty} (e_i + e_j)^T K_t(0)(e_i + e_j) = \lim_{t\to\infty} [TK_t(0)]_{ii} + TK_t(0)]_{ij} + TK_t(0)]_{ji} + TK_t(0)_{jj}] \tag{39}$$

$$= (e_i + e_j)^T K (e_i + e_j) = K_{ii} + K_{ij} + K_{ji} + K_{jj} \tag{40}$$

From eq.(38), we can observe that the diagonal elements of the matrix computed by the Riccati recursion converges to the diagonal elements of K. From eq.(40), we can observe that $TK_t(0)]_{ij} + TK_t(0)]_{ji}$ converges to $K_{ij} + K_{ji}$. Considering that $\{K_t(0)\}_{t\geq 0}$ is a sequence of symmetric matrices (which can be shown by induction using the Riccati recursion with zero initial matrix), $TK_t(0)]_{ij}$ and $TK_t(0)]_{ji}$ both converge to $(K_{ij} + K_{ji})/2$. (**Step 1** was proved).

Proof of Step 2: The closed-loop deterministic system can be expressed as

$$X_{t+1} = AX_t + BGX_t = DX_t \tag{41}$$

Per step cost can also be rewritten as

$$c(x_t, u_t) = x_t Q x_t + (Gx_t)^T R (Gx_t) = x_t (Q + G^T R G) x_t \tag{42}$$

In our case of average cost optimality, difference of average cost at time $t + 1$ ($X_{t+1} = x_{t+1}$) and time t ($X_t = x_t, U_t = Gx_t$) is the per-step cost function $c(x_t, Gx_t)$, and

$$x_{t+1}^T K x_{t+1} - x_t^T K x_t = x_t^T (D^T K D - K) x_t \tag{43}$$

$$= -x_t^T (Q + G^T R G) x_t < 0 \tag{44}$$

By applying the equation recursively,

$$x_{t+1}^T K x_{t+1} = x_t^T K x_t - x_t^T (Q + G^T R G) x_t \tag{45}$$

$$= x_{t-1}^T K x_{t-1} - \sum_{s=t-1}^{t} x_s^T (Q + G^T R G) x_s \tag{46}$$

$$= \ldots \tag{47}$$

$$= x_0^T K x_0 - \sum_{s=0}^{t} x_s^T (Q + G^T R G) x_s \tag{48}$$

In this expression, $x_0^T K x_0$ and $x_{t+1}^T K x_{t+1}$ are both nonnegative and $x_0^T K x_0$ is bounded (Recall **Claim a)-b)** of **Step 1**). Therefore,

$$\sum_{s=0}^{t} x_s^T (Q + G^T R G) x_s \tag{49}$$

is upperbounded and nondcreasing. This then gives

$$\lim_{t\to\infty} x_t^T (Q + G^T R G) x_t = 0 \tag{50}$$

To show that the state converges to 0 under the control $u = Gx$, we recall the assumptions that the matrices Q is positive semi-definite, R is positive definite, and $Q = C^T C$. Then, eq.(50) can be rewritten as

$$x_t^T C^T C x_t \xrightarrow{T\to\infty} 0 \tag{51}$$

$$x_t^T G^T R G x_t \xrightarrow{T\to\infty} 0 \tag{52}$$

or

$$C x_t \xrightarrow{T\to\infty} 0 \tag{53}$$

$$G x_t \xrightarrow{T\to\infty} 0 \tag{54}$$

Finally, **Step 2** can be proved by showing X_t converges to 0 as $t \to \infty$ using the assumption that the system is observable. For a fixed x_t, the state at time $t+k$ can be expressed as

$$\begin{aligned}
X_{t+k} &= A X_{t+k-1} + B U_{t+k-1} && (55)\\
&= A^2 X_{t+k-2} + B U_{t+k-1} + A B U_{t+k-2} && (56)\\
&= \dots && (57)\\
&= A^k X_t + \sum_{j=0}^{k-1} A^j B U_{t+k-1-j} && (58)\\
&= A^k X_t + \sum_{j=0}^{k-1} A^j B G X_{t+k-1-j} && (59)
\end{aligned}$$

, which gives

$$A^k X_t = X_{t+k} - \sum_{j=0}^{k-1} A^j B G X_{t+k-1-j} \tag{60}$$

By writing down the above equation for time $t, t+1, \dots, t+n-1$ in matrix form, we get

$$\begin{bmatrix} CA^{n-1} \\ \vdots \\ CA \\ C \end{bmatrix} X_t = \begin{bmatrix} C\left(X_{t+n-1} - \sum_{j=0}^{n-2} A^j B G X_{t+n-2-j}\right) \\ \vdots \\ C(X_{t+1} - BGX_t) \\ CX_t \end{bmatrix} \tag{61}$$

The observability matrix in the left hand side is full rank, and the right hand side converges to zero by equations (53-54). Therefore, X_t converges to 0 as $t \to \infty$ and the matrix $D = A + BG$ is stable.

Proof of Step 3: To show positive definiteness, for purpose of contradiction suppose there exists $x_0 \neq 0$ such that $x_0^\top K x_0 = 0$. Rewriting

$$x_{t+1}^\top K x_{t+1} = x_0^\top K x_0 - \sum_{s=0}^{t} x_s^\top (Q + G^\top RG) x_s,$$

we have that if $x_0^\top K x_0 = 0$, then $x_s^\top (Q + G^\top RG) x_s = 0,\ \forall s$. That, in turn, would mean that $x_s = 0$ (as $Q = C^\top C \geq 0$), and $Gx_s = 0$ (as $R \succ 0$). Together, these conditions imply $\begin{bmatrix} CA^{n-1} \\ \vdots \\ CA \\ C \end{bmatrix} x_0 = 0$, but then by observability (the matrix on LHS having full rank), x_0 would have to be 0, a contradiction.

Proof of Step 4-5: To show $K_t(K_0) \to K \equiv \lim_{t\to\infty} K_t(0),\ \forall K_0 = K_0^\top \succeq 0$, consider the deterministic control problem

$$\min_{u_0,\ldots,u_{t-1}} \sum_{s=0}^{t-1} x_s^\top Q x_s + u_s^\top R u_s \quad \text{s.t.}$$
$$x_{s+1} = Ax_s + Bu_s,\ x_0 \text{ given.}$$

Consider adding the terminal cost $x_t^\top K_t(K_0) x_t \geq x_0^\top K_t(0) x_0 \to x_0^\top K x_0$. For the other direction, substituting in $u_s = Gx_s$ and taking $t \to \infty$, by the stability of D, convergence is attained. As the upper and lower bounds meet, the limit exists. ■

9 The Kalman Filter

Eventually, we'll be looking at the LQG (linear quadratic Gaussian) problem with partial observations, described as

$$X_{t+1} = AX_t + BU_t + W_t \tag{62}$$
$$Y_t = CX_t + V_t \tag{63}$$
$$X_0, W_0, W_1, \ldots, V_0, V_1, \ldots \text{ independent} \tag{64}$$
$$X_0 \sim \mathcal{N}(0, \Sigma) \tag{65}$$
$$W_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \Sigma_W) \tag{66}$$
$$V_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \Sigma_V) \tag{67}$$
$$c(x, u) = x^\top Q x + u^\top R u \tag{68}$$
$$\bar{J}(x; g) = \lim \sum_{t\to\infty} \frac{1}{T} \mathbf{E}^g \Big(\sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t \Big| X_0 = x \Big). \tag{69}$$

However, we have access to only the observations, and thus must pass through the belief state $\pi_t(\cdot) = \mathbf{P}(X_t \in \cdot | Y_0^t, U_0^t)$. Gaussianity will enable us to obtain nice expressions for our quantities of interest.

Consider the Gaussian HMM specified above. Our goal is to recursively compute the Bayesian filter. Note that because $X_0, Y_0, X_1, Y_1, \dots$ are Gaussian, they're also jointly Gaussian. Hence

$$\pi_{t|s}(\cdot) := \mathbf{P}(X_t \in \cdot | Y_0^s) \equiv \mathcal{N}(\mu_{t|s}, K_{t|s}),$$

where $\mu_{t|s} := \mathbf{E}(X_t | Y_0^s)$ and $K_{t|s} := \text{Cov}(X_t | Y_0^s)$. In particular, $\pi_t(\cdot) = \mathcal{N}(\mu_{t|t}, K_{t|t}),\ \forall t$.

As an aside (further motivating the study of the Gaussian setting), note that keeping track of the belief state is hard, as it may have an arbitrarily complicated representation. In the Gaussian case, though, it's completely determined by the the conditional mean and variance, which have clean parametric forms.

Our updates follow the typical prediction-correction procedure,

$$\mu_{t|t} \xrightarrow{\text{pred}} \mu_{t+1|t} \xrightarrow{\text{corr}} \mu_{t+1|t+1}$$

and

$$K_{t|t} \xrightarrow{\text{pred}} K_{t+1|t} \xrightarrow{\text{corr}} K_{t+1|t+1}.$$

9.1 Facts about Gaussian estimation

We will make use of the following properties of Gaussian rv's in our analysis:

1. If X and Y are jointly Gaussian, then $\mathbf{E}(X|Y) = \mathbf{E}(X) + \text{Cov}(X,Y)\text{Cov}(Y)^{-1}(Y - \mathbf{E}(Y))$, and $\text{Cov}(X - \mathbf{E}(X|Y)) = \text{Cov}(X) - \text{Cov}(X,Y)\text{Cov}(Y)^{-1}\text{Cov}(X)^{-1}$.

2. If X, Y, and Z are jointly Gaussian and $Y \perp\!\!\!\perp Z$, then $\mathbf{E}(X|Y,Z) = \mathbf{E}(X|Y) + \text{Cov}(X,Z)\text{Cov}(Z)^{-1}(Z - \mathbf{E}(Z))$.

9.2 Analysis

Step 1:
The prediction calculations are $\mathbf{P}(X_{t+1} \in \cdot | X_t) \equiv \mathcal{N}(AX_t, \Sigma_W)$ and $\mathbf{P}(Y_{t+1} \in \cdot | Y_t) \equiv \mathcal{N}(CX_t, \Sigma_V)$, which are helped by the Gaussianity of the primitive rv's. In particular,

$$\begin{aligned} \mu_{t+1|t} &= \mathbf{E}(X_{t+1} | Y_0^t) \\ &= \mathbf{E}(AX_t + W_t | Y_0^t) \\ &= A\mathbf{E}(X_t | Y_0^t) + \mathbf{E}(W_t | Y_0^t) \\ &= A\mu_{t|t}, \end{aligned} \tag{70}$$

where in the last line, we used the fact that Y_0^t depends on $W_0^{t-1} \perp\!\!\!\perp W_t$.

Now define $\bar{X}_{t|s} := X_t - \mu_{t|s}$. This gives

$$\begin{aligned}\bar{X}_{t+1|t} &= X_{t+1|t} - \mu_{t+1|t} \\ &= AX_t + W_t - A\mu_{t|t} \\ &= A(X_t - \mu_{t|t}) + W_t \\ &= A\bar{X}_{t|t} + W_t,\end{aligned}$$

where $\bar{X}_{t|t} \perp\!\!\!\perp W_t$. Then, as $K_{t|t} = \mathbf{E}(\bar{X}_{t|t}\bar{X}_{t|t}^\top)$,

$$K_{t+1|t} = AK_{t|t}A^\top + \Sigma_W. \tag{71}$$

Step 2:
Define $\nu_{t|s} := \mathbf{E}(Y_t|Y_0^s)$, and $K_{t|s}^Y := \mathrm{Cov}(Y_t|Y_0^s)$. Then

$$\begin{aligned}\nu_{t|t-1} &= \mathbf{E}(Y_t|Y_0^{t-1}) \\ &= \mathbf{E}(CX_t + V_t|Y_0^{t-1}) \\ &= C\mu_{t|t-1},\end{aligned}$$

where the last line follows from the definition of $\mu_{t|s}$ and $V_t \perp\!\!\!\perp Y_0^{t-1}$; $\nu_{t|t-1}$ lets us compute

$$K_{t|t-1}^Y = CK_{t|t-1}C^\top + \Sigma_V. \tag{72}$$

Step 3:
We have $\mu_{t|t} = \mathbf{E}(X_t|Y_0^t) = \mathbf{E}(X_t|Y_0^{t-1}, Y_t)$. Now note that (Y_0^{t-1}, Y_t) and $(Y_0^{t-1}, \bar{Y}_{t|t-1})$ are probabilistically equivalent; $\bar{Y}_{t|t-1} = Y_t - \mathbf{E}(Y_t|Y_0^{t-1})$. However, the second representation is better for our purposes because $Y_0^{t-1} \perp\!\!\!\perp \bar{Y}_{t|t-1}$. Analysis

Using the facts about Gaussians given above, along with the expression for $K_{t|t-1}^Y$ from (72), we obtain

$$\begin{aligned}\mu_{t|t} &= \mathbf{E}(X_t|Y_0^{t-1}, \bar{Y}_{t|t-1}) \\ &= \mathbf{E}(X_t|Y_0^{t-1}) + \mathrm{Cov}(X_t|\bar{Y}_{t|t-1})\mathrm{Cov}(\bar{Y}_{t|t-1})^{-1}\bar{Y}_{t|t-1} \\ &= \mu_{t|t-1} + K_{t|t-1}C^\top(\Sigma_v + CK_{t|t-1}C^\top)^{-1}(Y_t - C\mu_{t|t-1})\end{aligned} \tag{73}$$

and

$$K_{t|t} = \mathrm{Cov}(\bar{X}_{t|t}Y_0^t) = K_{t|t-1} - K_{t|t-1}C^\top(\Sigma_V + CK_{t|t-1}C^\top)^{-1}CK_{t|t-1}. \tag{74}$$

We can then substitute in (74) into (71) and obtain

$$K_{t+1|t} = AK_{t|t}A^\top + \Sigma_W = A(K_{t|t-1} - K_{t|t-1}C^\top(\Sigma_V + CK_{t|t-1}C^\top)^{-1}CK_{t|t-1})A^\top + \Sigma_W. \tag{75}$$

Now if we examine the Kalman filter, we see that the problem of obtaining $K_{t+1|t}$ from $K_{t|t-1}$ is given by the Riccati recursion, with the following variable substitutions:

$$\begin{aligned}A &\leftrightarrow A^\top \\ C &\leftrightarrow B^\top \\ R &\leftrightarrow \Sigma_V \\ Q &\leftrightarrow \Sigma_W.\end{aligned}$$

If we keep running this Kalman filter, we arrive at a steady state, where the gain matrix is given by

$$\bar{K} := \lim_{t\to\infty} K_{t+1|t} = A(\bar{K} - \bar{K}C^\top(\Sigma_V + C\bar{K}C^\top)^{-1}C\bar{K})A^\top + \Sigma_W. \tag{76}$$

A unique pd solution $\bar{K}$ will exist if the following conditions are satisfied:

1. $(A^\top, C^\top)$ is controllable $\iff$ (A, C) is observable;
2. $\Sigma_V \succ 0$, i.e., the observation model is nondegenerate (geometrically, there exists noise in every direction); and
3. $(A^\top, \Gamma^\top)$ is observable $\iff$ (A, Γ) is controllable, where $\Sigma_W =: \Gamma\Gamma^\top$.

Note that our uncontrolled system, per (62) and (63) and the definition of Γ, is equivalently described as

$$X_{t+1} = AX_t + \Gamma\epsilon_t, \quad \epsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, I) \tag{77}$$

$$Y_t = CX_t + V_t. \tag{78}$$

Hence we are asking that (A, Γ) be controllable and (A, C) be observable, as well as $\mathrm{Cov}(V_t) = \Sigma_V \succ 0$.

Then $\bar{D}^\top = A^\top + C^\top\bar{G}$ will be stable, where

$$\bar{G} = -(\Sigma_V + C\bar{K}C^\top)^{-1}C\bar{K}A^\top,$$

so

$$\bar{D}^\top = A^\top - C^\top(\Sigma_V + C\bar{K}C^\top)^{-1}C\bar{K}A^\top.$$

As $\bar{D}$ has the same eigenstructure as $\bar{D}^\top$, $\bar{D} = A - A\bar{K}C^\top(\Sigma_V + C\bar{K}C^\top)^{-1}C$ is also stable. With some more work, we can then show that $\lim_{t\to\infty} \mathrm{Cov}(X_{t|t-1}) = \bar{K}$.

The takeaway is that the steady state of the Kalman filter is useful because the asymptotic behavior is shared by the time-varying (non-stationary) process.