Problems to be handed in

1 Consider the linear Itô SDE

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + \mathrm{d}W_t, \qquad t \ge 0 \tag{1}$$

where $W = \{W_t\}_{t\ge 0}$ is the standard n-dimensional Wiener process and $A \in \mathbb{R}^{n\times n}$ is a given matrix.

(i) Prove that the process $\{X_t\}$ can be written as

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}\,\mathrm{d}W_s.$$

Hint: Apply Itô's rule to $e^{-At}X_t$.

(ii) Prove that the process $\{X_t\}$ is Gaussian (i.e., for any finite collection of times $0 \le t_0 < t_1 < \ldots < t_m$, the random vectors $X_{t_0}, X_{t_1}, \ldots, X_{t_m}$ are jointly Gaussian).

(iii) Find the mean $\mathbf{E}[X_t]$ and the covariance matrix $\mathrm{Cov}(X_t) = \mathbf{E}[(X_t - \mathbf{E}[X_t])(X_t - \mathbf{E}[X_t])^\top]$. What can you say about the behavior of the process as $t \to \infty$?

(iv) Consider now the controlled SDE

$$\mathrm{d}X_t = (AX_t + BU_t)\,\mathrm{d}t + \mathrm{d}W_t, \qquad t \ge 0$$

where $B \in \mathbb{R}^{n\times m}$ is a given matrix and $U = \{U_t\}_{t\ge 0}$ is an $\{\mathcal{F}^X_t\}$-adapted input process on $\mathbb{R}^m$ (here, $\mathcal{F}^X_t = \sigma(X_s : 0 \le s \le t)$ is the σ-algebra generated by the trajectory of X up to time t). Prove that

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}U_s\,\mathrm{d}s + \int_0^t e^{A(t-s)}\,\mathrm{d}W_s.$$

2 So far, we have considered finite-horizon optimal control problems for diffusions. In this problem and the next one, we will obtain Bellman-type results for infinite-horizon problems. Consider a controlled diffusion process with state space $\mathcal{X} = \mathbb{R}^n$ and an arbitrary action space $\mathcal{U}$, specified by the controlled drift $b(x,u)$ and diffusion matrix $\sigma(x,u)$. Let $c : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}_+$ be a nonnegative and bounded cost function. Given a stationary feedback control law $g : \mathbb{R}^n \to \mathcal{U}$, consider the discounted cost

$$J_\lambda(g) := \mathbf{E}^g\left[\int_0^\infty e^{-\lambda t}c(X_t, U_t)\,\mathrm{d}t\right],$$

where $\lambda > 0$ is the exponential discount rate and the expectation is taken w.r.t. the diffusion process with drift $b^g(x) := b(x, g(x))$ and diffusion matrix $\sigma^g(x) := \sigma(x, g(x))$. Suppose that there exists a twice differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, such that $|\mathbf{E}[V(X_0)]| < \infty$ and

$$\min_{u\in\mathcal{U}}\left\{\mathcal{A}^u V(x) - \lambda V(x) + c(x,u)\right\} = 0$$

for all $x \in \mathbb{R}^n$. Let $\mathcal{G}_\lambda$ be the collection of all stationary control laws g, such that $e^{-\lambda t}\mathbf{E}^g[V(X_t)] \xrightarrow{t\to\infty} 0$. Show that the stationary feedback control

$$g^*(x) = \arg\min_{u\in\mathcal{U}} \left\{\mathcal{A}^u V(x) - \lambda V(x) + c(x,u)\right\}$$

belongs to $\mathcal{G}_\lambda$ and is optimal, i.e., for any $g \in \mathcal{G}_\lambda$,

$$J_\lambda(g) \ge J_\lambda(g^*) = \mathbf{E}[V(X_0)].$$

Hint: Consider any admissible g and apply Itô's rule to $e^{-\lambda t}V(X_t^g)$.

3 Now consider the setting of Problem 2, but with the long-term average-cost criterion

$$\bar{J}(g) := \limsup_{T\to\infty} \frac{1}{T}\mathbf{E}^g\left[\int_0^T c(X_t, U_t)\,\mathrm{d}t\right].$$

Suppose that there exists a twice differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and a constant $\eta \ge 0$, such that

$$\min_{u\in\mathcal{U}} \left\{\mathcal{A}^u V(x) + c(x,u)\right\} = \eta$$

for all $x \in \mathbb{R}^n$. Let $\bar{\mathcal{G}}$ denote the class of stationary feedback controls $g : \mathbb{R}^n \to \mathcal{U}$, such that

$$\limsup_{T\to\infty} \frac{\mathbf{E}[V(X_0) - V(X_t^g)]}{T} = 0.$$

Prove that

$$g^*(x) = \arg\min_{u\in\mathcal{U}} \left\{\mathcal{A}^u V(x) + c(x,u)\right\}$$

belongs to $\bar{\mathcal{G}}$ and is optimal, i.e., for any $g \in \bar{\mathcal{G}}$,

$$\bar{J}(g) \ge \bar{J}(g^*) = \eta.$$

Hint: Consider an arbitrary $g \in \bar{\mathcal{G}}$. Fix some finite horizon T, apply Itô's rule to $V(X_T^g)$, then exploit the Bellman equation and take the limsup.