Problems to be handed in

1 Give a direct proof of Theorem 10.2 in the lecture notes using dynamic programming.

2 Consider the steady-state Kalman filter, as presented in Section 10.3 of the lecture notes. Suppose the following conditions are met:

(i) (A, C) is an observable pair;

(ii) the observation model is nondegenerate, i.e., $\Sigma_V \succ 0$;

(iii) there exists a matrix Γ, such that $\Sigma_W = \Gamma\Gamma^\top$ and (A, Γ) is a controllable pair.

Prove that the steady-state Kalman filter is asymptotically optimal, i.e., that

$$\lim_{t\to\infty} \mathrm{Cov}(X_t - \widehat{X}_t) = (I - LC)K,$$

where K is the (unique, positive definite) solution of the discrete algebraic Riccati equation given in Eq. (57).

3 Let $\{X_t\}_{t\ge 0}$ be a continuous-time Markov process with finite state space $\mathcal{X} = \{1, \dots, n\}$ and transition intensities matrix $\Lambda = [\lambda_{i,j}]_{i,j\in\mathcal{X}}$. We can embed this process into $\mathbb{R}^n$ by representing each state $i \in \mathcal{X}$ by the ith canonical basis vector $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)^\mathsf{T} \in \mathbb{R}^n$, where only the ith coordinate is set to 1 and the rest are zero. Let $\{\tilde{X}_t\}_{t\ge 0}$ denote the resulting $\mathbb{R}^n$-valued Markov process with state space $\tilde{\mathcal{X}} := \{e_1, \dots, e_n\} \subset \mathbb{R}^n$.

(a) Prove that, for any $0 \le s < t$ and any $\tilde{x} \in \tilde{\mathcal{X}}$,

$$\mathbf{E}[\tilde{X}_t | \tilde{X}_s = \tilde{x}] = [e^{\Lambda(t-s)}]^\top \tilde{x}.$$

(b) For each $t \ge 0$, let $\mathcal{F}_t$ denote the σ-algebra generated by $\tilde{X}_{[0,t]} := \{\tilde{X}_s\}_{0\le s\le t}$. Consider the process $\{M_t\}$ defined by

$$M_t := \tilde{X}_t - \tilde{X}_0 - \int_0^t \Lambda^\top \tilde{X}_s \, \mathrm{d}s.$$

Prove that $\{M_t\}$ is an $\{\mathcal{F}_t\}$-martingale, i.e., for any $0 \le s < t$,

$$\mathbf{E}[M_t - M_s | \mathcal{F}_s] = 0.$$

4 In this problem, we will consider the simplest construction of a stochastic integral, due to Wiener, to illustrate the key ideas.

Let $W = (W_t)_{t\ge 0}$ be a standard one-dimensional Brownian motion. Our goal is to give meaning to an integral of the form

$$I(f) = \int_0^T f(t)\mathrm{d}W_t$$

for any *deterministic* function $f : [0,T] \to \mathbb{R}$ which is square-integrable, i.e.,

$$\|f\|_{L^2[0,T]} := \left(\int_0^T |f(t)|^2 \mathrm{d}t \right)^{1/2} < \infty.$$

(The extension to square-integrable adapted random processes is due to K. Itô, and bears his name.) Let $S[0,T]$ be the space of all *step functions* on the interval $[0,T]$ – that is, $f \in S[0,T]$ if and only if there exist finitely many time instants $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$, such that $f(t)$ is constant on each interval $[t_i, t_{i+1})$, $i = 0, \ldots, n-1$. For such an f, we *define*

$$I(f) := \sum_{i=0}^{n-1} f(t_i) \left(W_{t_{i+1}} - W_{t_i} \right).$$

Observe that, while f was deterministic, $I(f)$ is a *random variable*. Now, one of the basic results in functional analysis is that the step functions in $S[0,T]$ are *dense* in $L^2[0,T]$ (the Hilbert space of all square-integrable real-valued functions on $[0,T]$), i.e., for any $f \in L^2[0,T]$ we can find a sequence $\{f_n\}_{n=1}^\infty$ of step functions, such that $\|f_n - f\|_{L^2[0,T]} \xrightarrow{n\to\infty} 0$. Using this fact, we define the stochastic integral $I(f) = \lim_{n\to\infty} I(f_n)$. We will not worry about the details of this passage to the limit, but will instead investigate the properties of the integral $I(f)$ on $S[0,T]$.

(a) Prove the following properties:

(i) For any two $f_1, f_2 \in S[0,T]$ and any pair of coefficients c_1, c_2, $I(c_1 f_1 + c_2 f_2) = c_1 I(f_1) + c_2 I(f_2)$; moreover, the random variables $I(f)$, $f \in S[0,T]$, are jointly Gaussian.

(ii) For any two disjoint subintervals A, B of $[0,T]$, the random variables $I(\mathbf{1}_A)$ and $I(\mathbf{1}_B)$ are independent.

(iii) For any $f_1, f_2 \in S[0,T]$,

$$\mathbf{E}[I(f_1)I(f_2)] = \langle f_1, f_2 \rangle_{L^2[0,T]} := \int_0^T f_1(t) f_2(t) \mathrm{d}t,$$

and in particular $\mathbf{E}[I^2(f)] = \|f\|^2_{L^2[0,T]}$.

(iv) For any $f \in S[0,T]$, $I(f) \sim N(0, \|f\|^2_{L^2[0,T]})$.

(b) As noted earlier, all of these results continue to hold true for every $f \in L^2[0,T]$. With that in mind, prove the following: Let $\{\varphi_i\}_{i=1}^\infty$ be a complete orthonormal system of functions in $L^2[0,T]$, i.e., $\langle \varphi_i, \varphi_j \rangle_{L^2[0,T]} = \delta_{ij}$ for all i, j, where δ_{ij} is the Kronecker symbol. Then the random variables $I(\varphi_1), I(\varphi_2), \ldots$ are i.i.d. $N(0,1)$.