ECE 515 / ME 540 **Assignment # 5**

Issued: Oct 19 Due: Oct 26, 2023

Reading Assignment:

BMP, Ch. 6–7.

Problems:

1. Consider the LTI system $\dot{x} = Ax + Bu$. The *controllable subspace* $\mathcal{S}_c$ of this system (or of the pair (A, B)) is the range of its controllability matrix

$$\mathcal{C}(A, B) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix},$$

i.e., the span of the columns of $\mathcal{C}(A, B)$.

(i) Prove that the controllable subspace $\mathcal{S}_c$ is A-*invariant*, i.e., if $v \in \mathcal{S}_c$, then $Av \in \mathcal{S}_c$, and that it contains the range of B.

(ii) Let $\{v_1, \dots, v_{\bar{n}}\}$ be any basis of $\mathcal{S}_c$, where $\bar{n} = \operatorname{rank} \mathcal{C}(A, B)$, and let

$$V = \begin{pmatrix} v_1 & v_2 & \dots & v_{\bar{n}} \end{pmatrix}.$$

Use the results of (i) to prove that there exist matrices $\tilde{A}$ and $\tilde{B}$ of appropriate shapes, such that

$$AV = V\tilde{A} \qquad \text{and} \qquad B = V\tilde{B}.$$

(iii) In this part, we will assume that $\bar{n} < n$, i.e., the system is not controllable. Prove that there exists an invertible $n \times n$ matrix T, such that

$$AT = T\begin{pmatrix} \tilde{A} & A_{12} \\ 0 & A_{22} \end{pmatrix} \qquad \text{and} \qquad B = T\begin{pmatrix} \tilde{B} \\ 0 \end{pmatrix}$$

for some matrices A_{12}, A_{22} of appropriate shapes.

Hint: Take $T = \begin{pmatrix} V & U \end{pmatrix}$, where V is the matrix from part (ii) and where the columns of U are chosen in any way so that the columns of T are linearly independent.

(iv) Deduce the Kalman controllability canonical form from the result of (iii). In particular, prove that $(\tilde{A}, \tilde{B})$ is a controllable pair.

2. Consider a pair of matrices (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and let a matrix $K \in \mathbb{R}^{m\times n}$ be given. Prove that the controllability subspaces of (A, B) and $(A + BK, B)$ are equal, and thus $(A + BK, B)$ is controllable if and only if (A, B) is.

3. The system $\dot{x} = Ax + Bu; y = Cx$ is controlled using static output feedback $u = -Hy + v$. Show that the resulting closed-loop system

$$\dot{x} = (A - BHC)x + Bv; \qquad y = Cx$$

has the same controllability/observability properties as the original system.

Hint: use either the result of Problem 2 or the Hautus-Rosenbrock test.

4. Consider the system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u.$$

(i) Is this system asymptotically stabilizable by a state feedback $u = Kx$? If yes, find such a K; if not, explain why not.

(ii) Consider the output $y = (1\ 0\ 0)x$. Determine the transfer function of the resulting (open-loop) system *without directly computing it.* Explain your answer.

Hint: Recall the definition of a minimal realization.

5. Consider the system

$$\dot{x} = \begin{pmatrix} -1 & 0 & 4 \\ 2 & -2 & 3 \\ 0 & 1 & 1 \end{pmatrix} x, \quad y = (0 \quad 0 \quad 1)x$$

Design an asymptotic observer for x_1, x_2 in the form of a dynamical system with state dimension 2 (a *reduced-order* observer).