ECE 515 / ME 540 **Assignment # 4**

Issued: Oct 10 Due: Oct 18, 2023

Reading Assignment:

BMP, Ch. 4–6.

Problems:

1. Consider a system $\dot{x} = f(x)$. Suppose that $\frac{d}{dt}(x(t)^T Px(t)) \leq -x(t)^T Qx(t)$, where P and Q are symmetric positive definite matrices. Prove that, under this condition, the system is exponentially stable, in the sense that its solutions satisfy $|x(t)| \leq ce^{-\mu t}|x(0)|$ for some $c, \mu > 0$. Note that this statement is true whether the system is linear or not.

   Hint: use the inequality $\lambda_{\min}(M)|x|^2 \leq x^T Mx \leq \lambda_{\max}(M)|x|^2$, where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ are, respectively, the largest and the smallest eigenvalues of the matrix M. You may also use the fact that if a function $v(t)$ satisfies $\dot{v} \leq -av$, then $v(t) \leq e^{-at}v(0)$.

2. Show that, if all eigenvalues of a matrix A have real parts *strictly less* than some $-\mu < 0$, then for every $Q = Q^T > 0$ the equation $PA + A^T P + 2\mu P = -Q$ has a unique solution $P = P^T > 0$. Show that in this case the solutions of the LTI system $\dot{x} = Ax$ satisfy $|x(t)| \leq ce^{-\mu t}|x(0)|$ for some $c > 0$. (The number μ is called a *stability margin.*)

3. (corrected) Recall that, for an LTI system $\dot{x} = Ax + Bu$, the controllability Gramian has the form

   $$W(0, t) = \int_0^t e^{-A\tau} BB^T e^{-A^T\tau} d\tau.$$

   (i) Prove that the matrix

   $$\bar{W}(0, t) := \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau.$$

   is nonsingular for some $t > 0$ if and only if $W(0, t)$ is.

   (ii) The result of (i) implies that the pair (A, B) is controllable if and only if the pair $(-A, B)$ is controllable. Is this true for LTV systems? Prove or give a counterexample.

4. Consider the LTI system

   $$\dot{x} = Ax$$
   $$y = Cx$$

and suppose that the eigenvalues of A have negative real parts. Consider the function $V(x) = x^T M x$, where M denotes the observability Gramian for the *infinite* time horizon, i.e.,

$$M = M(0, \infty) = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$$

Show that, along the solutions of the system, we have

$$\dot{V} = -|y|^2.$$

5. Construct minimal (i.e., controllable and observable) realizations of the following transfer functions:

$$\frac{s-3}{s^2-5s+6}, \qquad \frac{s^2+1}{s^3-2s^2+s}.$$