ECE 515 / ME 540 **Assignment # 3**

Issued: Sep 14 Due: Sep 21, 2023

Reading Assignment:

BMP, Ch. 3; Ch. 4, Sec. 1–3.

Problems:

1. Let A be a 2×2 matrix with complex eigenvalues $\lambda_{1,2} = \sigma \pm j\omega$.

   (i) Show that A is similar to the matrix

   $$\bar{A} = \begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix},$$

   i.e., there exists an invertible 2×2 matrix P such that $A = P\bar{A}P^{-1}$.

   Hint: Use the Cayley–Hamilton theorem.

   (ii) Show that the matrix exponential $e^{t\bar{A}}$ is given by

   $$e^{t\bar{A}} = \begin{pmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{pmatrix} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

   Hint: Use the fact that $\bar{A} = \sigma I + \omega J$, where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

   (iii) Using the above results, compute the matrix exponential e^{tA} for

   $$A = \begin{pmatrix} 3 & -10 \\ 2 & -5 \end{pmatrix}.$$

2. (Exercise 3.9.2 in BMP) If A is an $n \times n$ matrix of full rank (i.e., it is invertible), show that

   $$\int_0^t e^{\tau A} d\tau = A^{-1}(e^{tA} - I_n), \qquad \forall t \in \mathbb{R}.$$

   Hint: This is easy once you realize that the two functions of t (one on the left-hand side, the other on the right-hand side) solve the same differential equation.

   Using the above result, obtain the solution to the linear time-invariant equation

   $$\dot{x} = Ax + B\bar{u}, \qquad x(0) = x_0$$

   where $\bar{u}$ is a constant m-dimensional vector and B is an $n \times m$ matrix.

3. (Exercise 3.9.7 in BMP) Suppose that A and B are constant square matrices. Consider the linear time-varying system described by

   $$\dot{x}(t) = e^{-tA} B e^{tA} x(t).$$

   Show that its state transition matrix is

   $$\Phi(t, s) = e^{-tA} e^{(A+B)(t-s)} e^{sA}.$$

4. Consider the system

$$\dot{x} = -\nabla V(x),$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a C^2 function (i.e., V and all its first- and second-order partial derivatives are continuous functions). Let $\bar{x}$ be a *stationary point* of V, i.e., $\nabla V(\bar{x}) = 0$. Find the conditions on V so that $\bar{x}$ is an asymptotically stable equilibrium for this system.

Hint: Try V as a candidate Lyapunov function.