ECE 515 / ME 540 **Assignment # 2**

Issued: Aug 31 Due: Sep 7, 2023

Reading Assignment:

BMP, Ch. 2.

Problems:

1. (Exercise 2.11.1 in BMP) Which of the following sets are fields? Justify your answers.

   (i) The set of integers with the usual operations.

   (ii) The set of rational numbers with the usual operations.

   (iii) The set of polynomials of degree less than 3 with real coefficients with addition and multiplication of polynomials.

   (iv) The set of all $n \times n$ nonsingular matrices with real-valued entries.

   (v) The set $\{0,\ 1\}$ with addition being binary "exclusive-or" and multiplication being binary "and".

2. Which of the following are vector spaces over $\mathbb{R}$ (with respect to the standard addition and scalar multiplication)? Justify your answers.

   (i) The set of real-valued $n \times n$ matrices with nonnegative entries, where n is a given positive integer.

   (ii) The set of rational functions of the form $\frac{q(s)}{p(s)}$, where q and p are polynomials in the complex variable s and the degree of p does not exceed a given fixed positive integer k.

   (iii) The space $L^2(\mathbb{R})$ of square-integrable functions, i.e., functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $\int_{-\infty}^{\infty} f^2(t)dt < \infty$.

3. Let A be the linear operator in the plane corresponding to the counter-clockwise rotation around the origin by some given angle θ. Compute the matrix of A relative to the standard basis in $\mathbb{R}^2$.

4. Let A be the linear operator from the previous problem. Compute the matrix of A relative to the basis $\left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

5. Let $A : V \to W$ be a linear operator between finite-dimensional vector spaces.

   (i) Prove that $\dim N(A) + \dim R(A) = \dim V$ (the sum of the dimension of the nullspace of A and the dimension of the range of A equals the dimension of V).

(ii) Now assume that $V = W$. It is *not* always true that V is a direct sum of $N(A)$ and $R(A)$. Find a counterexample demonstrating this. Also, describe a class of linear operators (as general as you can think of) for which this statement *is* true.

Reminder: Let W_1 and W_2 be subspaces of a vector space V. We say that V is a *direct sum* of W_1 and W_2, and write $V = W_1 \oplus W_2$, if $V = W_1 + W_2$ (i.e., any vector $v \in V$ can be represented as a sum $w_1 + w_2$ with $w_i \in W_i$, $i = 1, 2$) and $W_1 \cap W_2 = \{0\}$.