Infinite-Time Optimal Control

① LQR problem (cont.)

$x \in \mathbb{R}^n; \ u \in \mathbb{R}^m$

$\dot{x} = Ax + Bu \qquad t \geq 0$

$q(x,u) = u^T R u + x^T Q x$ where $R = R^T > 0$ in $\mathbb{R}^{m \times m}$, $Q = Q^T \geq 0$ in $\mathbb{R}^{n \times n}$

Goal:

$$\min \ J_\infty(x, u(\cdot)) = \int_0^\infty \{u(t)^T R u(t) + x(t)^T Q x(t)\} dt$$

s.t. $\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x$

Bellman (value) fcn: $V(t,x) := \min\limits_{u(\cdot)} \int_t^\infty q(x(s), u(s)) ds$

s.t. $\dot{x}(s) = Ax(s) + Bu(s), \ s \geq t$
$x(t) = x$

$\Rightarrow \ V(x) := V(0,x) = \min\limits_{u(\cdot)} J_\infty(x, u(\cdot))$

Thm (LQR: optimal control)

Assume that (A,B) is a controllable pair. Then there exists a solution $\Pi = \Pi^T \geq 0$ of the Algebraic Riccati Equation (ARE)

$$\Pi B R^{-1} B^T \Pi - \Pi A - A^T \Pi - Q = 0,$$

such that the following holds:

1) $V(x) = x^T \Pi x$

2) the state feedback law $k(x) = -Fx$, where $F := R^{-1} B^T \Pi$, is optimal. Moreover, under this control,

$$\lim_{t \to \infty} x(t)^T \Pi x(t) = 0.$$

3) If, in addition, $Q = Q^T > 0$, then $x(t) \to 0$ as $t \to \infty$ from any i.c. under $k(x) = -Fx$.

Proof (cont. from last lecture)

Let $(x(t), \bar{u}(t))$ be given by

$$\dot{x}(t) = Ax(t) + Bk(x(t)), \quad \bar{u}(t) = k(x(t)) = Fx(t)$$
$$x(0) = x$$

where $F = R^{-1}B^T\Pi$. C.l. system: $\dot{x} = (A+BF)x$

$$J_\infty(x, \bar{u}(\cdot)) = \int_0^\infty \{(Fx(t))^T RFx(t) + x(t)^T Q x(t)\} dt$$

$$= \int_0^\infty x(t)^T \{\Pi B R^{-1} B^T \Pi + Q\} x(t)\, dt$$

Because Π solves the ARE, it can be shown that

$$\frac{d}{dt} x(t)^T \Pi x(t) = -x(t)^T \{\Pi B R^{-1} B^T \Pi + Q\} x(t)$$

$\Rightarrow$ for $\bar{u}(\cdot)$,

$$J_\infty(x, \bar{u}(\cdot)) = -\int_0^\infty \frac{d}{dt} x(t)^T \Pi x(t)\, dt$$

$$= x(0)^T \Pi x(0) - \lim_{t\to\infty} x(t)^T \Pi x(t)$$

$$= x^T \Pi x - \lim_{t\to\infty} x(t)^T \Pi x(t)$$

$$\leq x^T \Pi x$$

$$= \lim_{t\to\infty} x^T \Pi(t) x$$

$\Pi(t)$ is the Bellman fcn for the optimal LQR problem on $[0,t]$ with $q(x,u) = u^T R u + x^T Q x$ and $p(x) = 0$

$$\leq J_\infty(x, v(\cdot)) \quad \forall\, v(\cdot)$$

$$\Rightarrow \quad J_\infty(x, \bar{u}(\cdot)) = \min_{v(\cdot)} J_\infty(x, v(\cdot)) = V(x)$$

$$\lim_{t\to\infty} x(t)^T \Pi x(t) = 0.$$

This proves 1), 2).

Now assume $Q = Q^T > 0$. We will show that, in this case, $\Pi > 0$, so $x(t) \to 0$ as $t \to \infty$ under the c.l. system dynamics $\dot{x} = (A + BF)x$.

Recall that $\Pi = \lim_{t \to \infty} \Pi(t)$, where

$$\dot{\Pi}(t) = -\Pi(t) B R^{-1} B^T \Pi(t) + \Pi(t) A + A^T \Pi(t) + Q$$

$$t \geq 0, \qquad \Pi(0) = 0.$$

Claim: $\Pi(t) > 0, \ \forall t.$ $\left[t \mapsto x^T \Pi(t) x \text{ is nondecreasing} \right]$

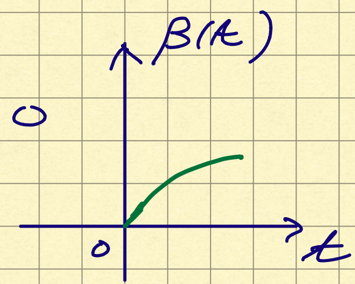
Fix $x \in \mathbb{R}^n \setminus \{0\}$

$$\beta(t) := x^T \Pi(t) x$$

$$\beta(0) = x^T \Pi(0) x = 0$$

$$\dot{\beta}(0) = x^T \dot{\Pi}(0) x = x^T Q x > 0$$

so $\beta(t) > 0$ for small $t > 0$



But $\beta(t)$ is nondecreasing, so $\beta(t) > 0 \ \forall t > 0$, i.e. $\Pi(t) = \Pi(t)^T > 0$ for all t.

$\Pi = \lim_{t \to \infty} \Pi(t) > 0$ (because entries of $\Pi(t)$ are nondecreasing)

Since $\Pi > 0$, $x(t)^T \Pi x(t) \to 0$ as $t \to \infty$
$\iff x(t) \to 0$ as $t \to \infty$. ∎

Aside: if (A, B) is a controllable pair, then $\exists$ matrix F s.t. $A + BF$ is Hurwitz.

② Nonlinear Stabilizing Optimal Control

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m$$

For each $x \in \mathbb{R}^n$, define $\mathcal{U}_x$ (the set of stabilizing controls):

$u : [0,\infty) \to \mathbb{R}^m$ s.t.

$$\dot{x}(t) = f(x(t), u(t))$$
$$x(0) = x$$

has a solution for all $t \geq 0$ $\Leftrightarrow$ $\underbrace{\phi(t, x, u(\cdot))}_{:= x(t)}$ is well-defined

and $x(t) \to 0$ as $t \to \infty$.

Objective: let $x \in \mathbb{R}^n$ (i.c.) be given;

$$\min \quad J_\infty(x, u(\cdot)) := \int_0^\infty q(x(t), u(t))\, dt$$

over all stabilizing controls $u(\cdot) \in \mathcal{U}_x$.

Main idea: Bellman/Lyapunov fcns

Given $V : \mathbb{R}^n \to \mathbb{R}$ C^1
$V(0) = 0$

Define $\dot{V} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ by

$$\dot{V}(x,u) := \frac{\partial V}{\partial x}(x) f(x,u) \equiv \nabla V(x)^T f(x,u)$$

Proposition Assume:

1) $\dot{V}(x,u) + q(x,u) \geq 0 \quad \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m$

2) $\exists\, u(\cdot) \in \mathcal{U}_{x^0}$ s.t.

$$\dot{V}(x(t), u(t)) + q(x(t), u(t)) = 0, \quad \text{for almost all } t \geq 0$$

Then $V(x^0)$ is the Bellman fcn, and $u(\cdot)$ is an optimal control:

$$V(x^0) = J_\infty(x^0, u(\cdot)) = \min_{v(\cdot) \in \mathcal{U}_x} J_\infty(x^0, v(\cdot)).$$

Proof Fix any $v(\cdot) \in \mathcal{U}_{x^0}$ (assume nonempty)

$\xi(t) := \phi(t, x^0, v(\cdot))$ $\quad$ $\dot{\xi}(t) = f(\xi(t), v(t))$
$\xi(0) = x^0, \ t \geq 0$

$$V(\xi(t)) - V(x^0) = \int_0^t \frac{d}{ds} V(\xi(s))\,ds$$
$$= \int_0^t \dot{V}(\xi(s), v(s))\,ds$$
$$\geq -\int_0^t q(\xi(s), v(s))\,ds$$

$$\Rightarrow V(x^0) \leq V(\xi(t)) + \int_0^t q(\xi(s), v(s))\,ds, \qquad \forall t \geq 0$$

Since $v(\cdot) \in \mathcal{U}_{x^0}$, and V is cont.,
$$\lim_{t\to\infty} V(\xi(t)) = V(0) = 0,$$

thus $V(x^0) \leq \lim_{t\to\infty} \int_0^t q(\xi(s), v(s))\,ds \equiv J_\infty(x^0; v(\cdot))$

But if $v(\cdot) = u(\cdot)$ s.t. $\dot{V}(x(t), u(t)) + q(x(t), u(t)) = 0$ a.e.

s.t. $\dot{x}(t) = f(x(t), u(t))$, $x(0) = x^0$, then
$V(x^0) = J_\infty(x^0; u(\cdot))$. ∎

<u>Corollary</u> (infinite-time HJB equation)

Suppose $k: \mathbb{R}^n \to \mathbb{R}^m$ is s.t.

$$\boxed{\dot{V}(x, k(x)) + q(x, k(x)) = \min_{u\in\mathbb{R}^m}\{\dot{V}(x,u) + q(x,u)\} \equiv 0}$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

Then $(x(t), u(t))$ s.t. $\dot{x}(t) = f(x(t), k(x(t)))$
$u(t) = k(x(t))$
$x(0) = x^0$

and $u(\cdot) \in \mathcal{U}_{x^0}$, then: $V(x^0)$ is the optimal cost and $u(\cdot)$ is the optimal control.

Thm Let $k: \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz, s.t. HJB holds.

Suppose $V: \mathbb{R}^n \to \mathbb{R}$ is C^1, positive definite, and proper (has compact level sets).

Suppose also $q(x,u) > 0$ whenever $x \neq 0$ (for all u).

Then, for all x^0, the sol'n to

$$\dot{x} = f(x, k(x))$$
$$x(0) = x^0$$

exists for all $t \geq 0$; the control $u(t) = k(x(t))$ is optimal, and $V(x^0)$ is the Bellman fcn. Moreover, $V(\cdot)$ is the Lyapunov fcn for the closed-loop system.

Proof $\dot{V}(x, k(x)) = -q(x, k(x)) < 0$ for $x \neq 0$

thus, by Lyapunov stability, $x(t) \to 0$ as $t \to \infty$.
$\Rightarrow$ $u(t) = k(x(t)) \in \mathcal{U}_{x^0}$. ∎

Preview: (next lecture)

$$\dot{x} = f(x) + G(x)u \qquad \text{(control-affine)}$$

$$q(x,u) = u^T R(x) u + Q(x)$$

where $R(x) = R(x)^T > 0$ in $\mathbb{R}^{m \times m}$
$Q(x) \geq 0$, continuous in x