Optimal Control

Focus so far: stabilization, controllability, etc.

Optimality: accomplishing these tasks in "best possible way" (according to some criterion)

General setting:

System: $\dot{x} = f(t, x, u)$, $\quad x(t_0) = x \qquad x(t) \in \mathbb{R}^n$

$t_0 \le t \le t_1 \qquad u(t) \in \mathbb{R}^m$

Control law (input) $\quad u : [t_0, t_1] \to \mathbb{R}^m$

Cost : $\quad J(t_1, t_0, x, u(\cdot)) \leftarrow \min$

- instantaneous cost: $\quad q(t, x, u) \qquad t_0 \le t < t_1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$
- terminal cost: $\quad p(x), \quad x \in \mathbb{R}^n$

$$J(t_1, t_0, x, u(\cdot)) := \int_{t_0}^{t_1} q(t, x(t), u(t))\, dt + p(x(t_1))$$

subject to $\quad \dot{x}(t) = f(t, x(t), u(t))$

$x(t_0) = x, \qquad t \in [t_0, t_1]$

Optimal Control Problem:

$$\min J(t_1, t_0, x, u(\cdot)) \quad \text{over} \quad u(\cdot) \in \mathcal{U},$$

where $\mathcal{U}$ is a given class of admissible controls.

Example (to keep in mind): Linear Quadratic Regulator (LQR)

$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad$ (LTV)

$q(t, x, u) = u^T R(t) u + x^T Q(t) x \qquad R(t) = R(t)^T > 0$

$Q(t) = Q(t)^T \geq 0$

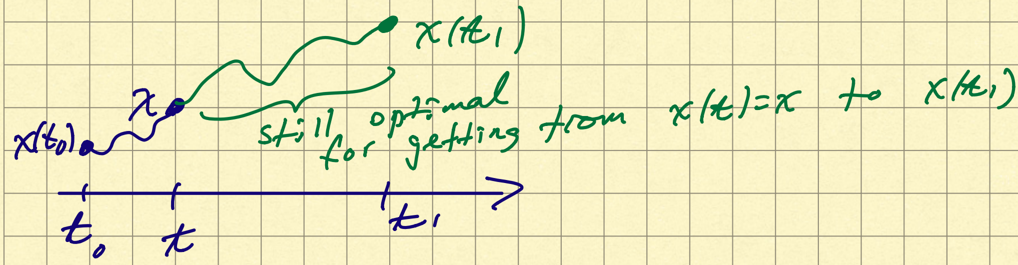
$p(x) = x^T S x \qquad S = S^T \geq 0$, constant

Cost to be minimized:

$$J(t_1, t_0, x, u(\cdot))$$
$$= \int_{t_0}^{t_1} \{u(t)^T R(t) u(t) + x(t)^T Q(t) x(t)\} dt + x(t_1)^T S x(t_1)$$

$$\text{s.t.} \quad \dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$x(t_0) = x, \quad t_0 \le t \le t_1$$

Dynamic Programming framework (R. Bellman)



Cost-to-go: $t_0 \le t \le t_1$, x, $u(\cdot)$:

$$J(t_1, t, x, u(\cdot)) := \int_t^{t_1} q(s, x(s), u(s))ds + p(x(t_1))$$

$$\text{s.t.} \quad \dot{x}(s) = f(s, x(s), u(s)), \qquad t \le s \le t_1$$
$$x(t) = x$$

Bellman fcn (value fcn, optimal cost-to-go):

$$V(t,x), \qquad t \in [t_0, t_1], \qquad x \in \mathbb{R}^n$$

$$V(t,x) := \inf_{u(\cdot)} J(t_1, t, x, u(\cdot))$$

Assumption: the Bellman fcn is achieved, i.e.

$\forall t \in [t_0, t_1), \forall x \in \mathbb{R}^n$: $\exists\, u^*(\cdot)$ s.t. $V(t,x) = J(t_1, t, x, u^*(\cdot))$

Dynamic Programming Lemma

Notation: • $\varphi(s,t,x,u(\cdot)) := x(s)$ for $t_0 \le t \le s \le t_1$, $x \in \mathbb{R}^n$, $u(\cdot)$ - adm. control

s.t. $\dot{x}(r) = f(r, x(r), u(r))$ $\quad s \geq r \geq t$

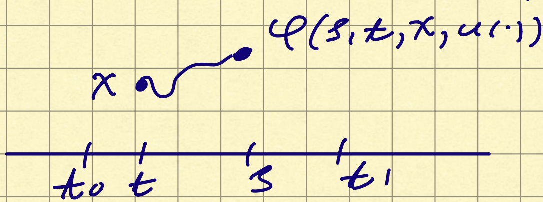
$x(t) = x$

• $Q(s,t,x,u(\cdot)) := \int_t^s q(r, x(r), u(r))\,dr$ starting at $x(t)=x$

Two key properties of V:

1) for all $t_0 \le t \le s \le t_1$, all $x \in \mathbb{R}^n$, all $u(\cdot)$,

$$V(t,x) \le Q(s,t,x,u(\cdot)) + V(s, \varphi(s,t,x,u(\cdot)))$$



2) If the Bellman fcn is achieved, then

$$V(t,x) = Q(s,t,x,u_{[t,s]}) + V(s, \varphi(s,t,x,u(\cdot))),$$

where $u(\cdot)$ is an optimal strategy that achieves V.

Implications: • $V(t_1, x) = p(x)$ $\quad \forall x \in \mathbb{R}^n$

$\vdots$

$$V(t_0, x) = \inf_{u(\cdot)} J(t_1, t_0, x, u(\cdot))$$

• Verification lemma: any $\tilde{V}(t,x)$ s.t.

0) $\tilde{V}(t_1, x) = p(x)$

1) $\tilde{V}(t,x) \le Q(s,t,x,u(\cdot)) + \tilde{V}(s, \varphi(s,t,x,u(\cdot)))$
for all $t_0 \le t \le s \le t_1$, x, $u(\cdot)$

2) $\exists$ a control $\bar{u}(\cdot)$ s.t.

$$\tilde{V}(t,x) = Q(s,t,x,\bar{u}(\cdot)) + \tilde{V}(s, \varphi(s,t,x,\bar{u}(\cdot)))$$

for all $t_0 \leq t \leq s \leq t_1$, $x \in \mathbb{R}^n$,

then $\tilde{V} = V$, and $\bar{u}(\cdot)$ is an optimal control.

Hamilton-Jacobi-Bellman (HJB) equation

$$\dot{x} = f(t,x,u)$$

$$J(t_1,t,x,u(\cdot)) = \int_t^{t_1} q(s,x(s),u(s))\,ds + p(x(t_1))$$

Let $u(\cdot)$ be an optimal control; then

$$V(t,x(t)) = \int_t^s q(r,x(r),u(r))\,dr + V(s,x(s))$$

$$\frac{d}{ds} V(s,x(s)) = -q(s,x(s),u(s))$$ [along the optimal trajectory]

$$\frac{d}{ds} V(s,x(s)) = \frac{\partial}{\partial s} V(s,x(s)) + \frac{\partial V}{\partial x}(s,x(s))\,\dot{x}(s)$$

$$-q(s,x(s),u(s)) = \frac{\partial}{\partial s} V(s,x(s)) + \frac{\partial V}{\partial x}(s,x(s))\,f(s,x(s),u(s))$$

Let $s \downarrow t$, $x(s) \downarrow x$ (given $x(t)=x$)

$$\boxed{\frac{\partial}{\partial t} V(t,x) = -q(t,x,u(t)) - \frac{\partial V}{\partial x}(t,x)\,f(t,x,u(t))}$$

$t_0 \leq t \leq t_1$, $V(t_1,x) = p(x)$

Let $v(\cdot)$ be another admissible control; then

$$V(t,x) \leq \int_t^s q(r,x(r),v(r))\,dr + V(s,x(s))$$

where $\dot{x}(r) = f(r,x(r),v(r))$

$x(t)=x$, $t \leq r \leq s$

$$\Rightarrow \frac{d}{dt} V(t, x \quad) \geq -q(t, x, v(t))$$

or $$\frac{\partial}{\partial t} V(t,x) \geq -\frac{\partial}{\partial x} V(t,x) f(t,x,v(t)) - q(t,x,v(t))$$

for all other $v(\cdot) \neq u(\cdot)$

Thus, we "obtain" the HJB equation

$$\frac{\partial}{\partial t} V(t,x) = -\min_{u \in \mathbb{R}^m} \left\{ q(t,x,u) + \frac{\partial}{\partial x} V(t,x) f(t,x,u) \right\}$$

for all $x \in \mathbb{R}^n$, $t \in [t_0, t_1]$

s.t. $V(t_1, x) = p(x)$

<u>Thm</u> If there exists a C^1 fcn $V(t,x)$ that solves the HJB equation, then:

$$V(t_0, x) = \min_{u(\cdot)} J(t_1, t_0, x, u(\cdot))$$

for all $x \in \mathbb{R}^n$ and the optimal control is given by $u(t) = k(t,x)$, where

$$k(t,x) = \operatorname*{argmin}_{u \in \mathbb{R}^m} \left\{ q(t,x,u) + \frac{\partial}{\partial x} V(t,x) f(t,x,u) \right\}.$$

<u>Note:</u> implicitly assuming uniqueness:

$$q(t,x,u) + \frac{\partial}{\partial x} V(t,x) f(t,x,u)$$

$$> q(t,x,k(t,x)) + \frac{\partial}{\partial x} V(t,x) f(t,x,k(t,x))$$

for all $u \neq k(t,x)$.

Back to LQR: $\dot{x}(t) = A(t)x(t) + B(t)u(t)$

$$J(t_1, t_0, x, u(\cdot)) = \int_{t_0}^{t_1} \{u(t)^T R(t) u(t) + x(t)^T Q(t) x(t)\} dt + x(t_1)^T S x(t_1)$$

$$V(t, x) := \min_{u(\cdot)} J(t_1, t, x, u(\cdot))$$

Thm (LQR optimality). Fix $t_0 < t_1$. Then (under mild regularity conditions) $\exists\, P(t) = P(t)^T$ in $\mathbb{R}^{n \times n}$ that solves the Riccati Diff. Eq. (RDE)

$$\dot{P} = PBR^{-1}B^T P - PA - A^T P - Q \qquad t_0 \leq t \leq t_1$$
$$P(t_1) = S$$

Let $F(t) := -R(t)^{-1} B(t)^T P(t)$. Then the feedback law $k(t, x) = F(t)x$ is optimal, and

$$V(t, x) = x^T P(t) x.$$

In particular, $V(t_0, x) = x^T P(t_0) x$ is the opt. value of the LQR problem.

Proof sketch: next lecture.