Online Parameter Estimation for LTI systems

Recap: $\dot{x} = Ax + bu$ $\quad y, u \in \mathbb{R}$ $\quad$ (SISO)

$y = c^T x$ $\quad x \in \mathbb{R}^n$

$A \in \mathbb{R}^{n \times n}, \; b \in \mathbb{R}^n, \; c \in \mathbb{R}^n$

i/o description:

$$y(t) = c^T e^{At} x(0) + \int_0^t c^T e^{A(t-s)} b u(s) ds, \quad t \geq 0$$

linear parametrization:

$$Y(s) = G(s) U(s) \qquad \text{(freq. domain)}$$

$$G(s) = c^T (sI_n - A)^{-1} b$$

$$= \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

nth-order controlled ODE (output y is controlled by input u)

$$(s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0) Y(s)$$

$$= (b_{n-1}s^{n-1} + \dots + b_1 s + b_0) U(s) \qquad [FD]$$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y$$

$$= b_{n-1} u^{(n-1)} + \dots + b_1 u' + b_0 u \qquad (TD]$$

Linear parametrization:

$$\boxed{y^{(n)} = \theta^T v}$$

where $\theta := (b_{n-1}, \dots, b_1, b_0, a_{n-1}, \dots, a_1, a_0)^T$ is the vector of sys. parameters

$v(t) := (u^{(n-1)}(t), \dots, u'(t), u(t), -y^{(n-1)}(t), \dots, -y'(t), -y(t))^T$

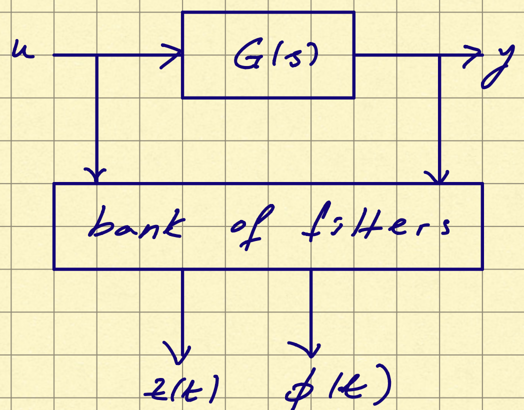
$$s^n Y(s) = \theta^T V(s)$$

$$Z(s) := \frac{s^n}{\Lambda(s)}, \qquad \Phi(s) = \left(\frac{s^{n-1}}{\Lambda(s)} U(s), \dots, \frac{1}{\Lambda(s)} U(s), -\frac{s^{n-1}}{\Lambda(s)} Y(s), \dots, -\frac{1}{\Lambda(s)} Y(s) \right)^T$$

where $\Lambda(s) = s^n + \lambda_{n-1} s^{n-1} + \dots + \lambda_1 s + \lambda_0$ is a polynomial with stable roots :

$\frac{s^n}{\Lambda(s)}, \frac{s^{n-1}}{\Lambda(s)}, \dots, \frac{s}{\Lambda(s)}, \frac{1}{\Lambda(s)}$ are stable filters

Back to time domain: $\boxed{z(t) = \theta^T \phi(t)}$



$G(s) = c^T (sI_n - A)^{-1} b$

$\longleftarrow$ do not depend on A, b, c

$z(t) = \theta^T \phi(t), \quad \theta \in \mathbb{R}^{2n}$

$\phi(t)$ is the vector of regressors

Upshot: we can process u, y in an online manner to get z, ϕ and estimate θ online using (ϕ, z) as data.

$$J_t(\hat{\theta}) := \frac{1}{2}(\hat{z}(t) - z(t))^2, \qquad \hat{z}(t) = \hat{\theta}^T \phi(t)$$

(instantaneous cost)

$$\bar{J}_t(\hat{\theta}) := \frac{1}{2}(\hat{\bar{z}}(t) - \bar{z}(t))^2 \qquad \bar{z}(t) = \theta^T \bar{\phi}(t)$$

$$\hat{\bar{z}}(t) = \hat{\theta}^T \bar{\phi}(t)$$

$$\bar{\phi}(t) := \frac{\phi(t)}{m(t)}$$

where $m(t)$ is a positive signal chosen so that $\frac{\phi}{m} \in \mathcal{L}_\infty$ e.g. $m(t) = 1 + \phi(t)^T \phi(t)$.

Gradient flow: $\dot{\hat{\theta}} = -\Gamma \nabla \bar{J}_t(\hat{\theta})$

where $\Gamma = \Gamma^T > 0$ is a fixed adaptation gain matrix

$$\bar{e}(t) := \hat{\bar{z}}(t) - \bar{z}(t)$$

$$= (\hat{\theta}(t) - \theta)^T \bar{\phi}(t)$$ — (normalized) prediction error

Theorem (Ioannou-Sun)

(i) $\hat{\theta} \in L_\infty$ and $\bar{e}, \dot{\hat{\theta}} \in L_2 \cap L_\infty$

(ii) if $\bar{\phi} = \frac{\phi}{m}$ is Persistently Exciting$^{(*)}$, then

$$\hat{\theta}(t) \to \theta \quad \text{as } t \to \infty$$

exponentially fast.

Remarks:

1) $\hat{\theta} \in L_\infty \iff \sup_{t \geq 0} |\hat{\theta}(t)| < \infty$

(parameter estimates don't escape to ∞)

2) $\bar{e}, \dot{\hat{\theta}} \in L_2 \cap L_\infty$

$$\bar{e}(t) := \hat{\bar{z}}(t) - \bar{z}(t)$$

$$= \underbrace{(\hat{\theta}(t) - \theta)^T}_{\tilde{\theta}(t)} \bar{\phi}(t) \qquad \bar{\phi} = \frac{\phi}{m} \in L_\infty$$

$\tilde{\theta}(t)$ — param. estimation error

$$\sup_{t \geq 0} |\bar{e}(t)| < \infty \qquad (\bar{e} \in L_\infty)$$

$$\int_0^\infty \bar{e}(t)^2 \, dt < \infty \qquad (\bar{e} \in L_2)$$

Same applies to $\dot{\hat{\theta}}$ (sensitivity to data)

$\dot{\hat{\theta}} \in L_\infty$ (bdd rate of change of $\hat{\theta}$)

$\dot{\hat{\theta}} \in L_2$ (slow adaptation)

3) PE for vector-valued signals

$$\bar{J}_t(\hat{\theta}) = \frac{1}{2}(\hat{\theta} - \theta)^T \bar{\phi}(t) \bar{\phi}(t)^T (\hat{\theta} - \theta)$$

$$\nabla \bar{J}_t(\hat{\theta}) = \bar{\phi}(t) \bar{\phi}(t)^T (\hat{\theta} - \theta)$$

$$\nabla^2 \bar{J}_t(\hat{\theta}) = \bar{\phi}(t) \bar{\phi}(t)^T$$ – rank-1 positive semidefinite

– $\nabla^2 \bar{J}_t$ is a singular matrix $\forall t \geq 0$

Integrated cost:

$$\bar{\bar{J}}_t(\hat{\theta}) = \int_0^t \bar{J}_s(\hat{\theta})\, ds$$

$$= \frac{1}{2}(\hat{\theta} - \theta)^T \int_0^t \bar{\phi}(s)\bar{\phi}(s)^T ds \,(\hat{\theta} - \theta)$$

$$\nabla^2 \bar{\bar{J}}_t(\hat{\theta}) = \int_0^t \bar{\phi}(s)\bar{\phi}(s)^T ds$$

– can be nonsingular if $\bar{\phi}$ persistently moves around

Def: $\bar{\phi}$ is PE if $\exists\, \alpha_0, T_0 > 0$ s.t.

$$\int_{t_0}^{t_0+T_0} \bar{\phi}(t)\bar{\phi}(t)^T dt \geq \alpha_0 T_0 I, \quad \forall t_0 \geq 0$$

or

$$\int_{t_0}^{t_0+T_0} v^T \bar{\phi}(t)\bar{\phi}(t)^T v \, dt \geq \alpha_0 T_0 |v|^2,$$

for all $t_0 \geq 0$ and all $v \in \mathbb{R}^{2n}$.

$$\dot{\hat{\theta}} = -\Gamma \nabla \bar{J}_t(\hat{\theta}) \qquad \tilde{\theta} = \hat{\theta} - \theta \qquad \dot{\tilde{\theta}} = \dot{\hat{\theta}}$$

$\Gamma = \Gamma^T > 0$ $\qquad$ Γ^{-1} exists

Proof

(i) Introduce candidate Lyapunov fcn

$$V(\tilde{\theta}) := \frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta}$$

$$\begin{aligned}
\dot V &= \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \qquad\qquad (\Gamma^{-1} = (\Gamma^{-1})^T > 0) \\
&= \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} \\
&= -\tilde{\theta}^T \Gamma^{-1} \Gamma \nabla \bar{J}_t(\hat{\theta}) \\
&= -\tilde{\theta}^T \nabla \bar{J}_t(\hat{\theta})
\end{aligned}$$

$$\begin{aligned}
\nabla \bar{J}_t(\hat{\theta}) &= \bar{\phi}\bar{\phi}^T(\hat{\theta} - \theta) \\
&= \underbrace{(\hat{\theta} - \theta)^T \bar{\phi}}_{\bar{e}}\, \bar{\phi}
\end{aligned}$$

$$\begin{aligned}
\bar{e}(t) &:= \hat{\bar{z}}(t) - \bar{z}(t) \\
&= (\hat{\theta}(t) - \theta)^T \bar{\phi}(t) \\
&= \tilde{\theta}(t)^T \bar{\phi}(t)
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\dot V &= -\tilde{\theta}^T \bar{\phi}\, \bar{e} \\
&= -\bar{e}^2 \le 0
\end{aligned}$$

So: $V(\tilde{\theta}(t))$ is nonnegative, decreasing

$$0 \le V(\tilde{\theta}(t)) \le V(\tilde{\theta}(0))$$

$$\frac{1}{2}\tilde{\theta}(t)^T \Gamma^{-1} \tilde{\theta}(t) \le \frac{1}{2}\tilde{\theta}(0)^T \Gamma^{-1} \tilde{\theta}(0)$$

$$\sup_{t \ge 0} |\tilde{\theta}(t)| < \infty \quad \Rightarrow \quad \tilde{\theta} \in L_\infty \quad \Rightarrow \quad \boxed{\hat{\theta} \in L_\infty}$$

Now,

$$\begin{aligned}
0 \le \int_0^t \bar{e}^2(s)\,ds &= -\int_0^t \dot V(\tilde{\theta}(s))\,ds \\
&= V(\tilde{\theta}(0)) - V(\tilde{\theta}(t)) \\
&\le V(\tilde{\theta}(0)) < \infty \qquad \forall t \ge 0
\end{aligned}$$

$\Rightarrow \bar{e} \in L_2$

$$\bar{e} = \tilde{\theta}^T \bar{\phi} \qquad |\bar{e}| \le \overset{L_\infty}{|\tilde{\theta}|}\,\overset{L_\infty}{|\bar{\phi}|} \quad \Rightarrow \quad \bar{e} \in L_\infty$$

$$\boxed{\bar{e} \in L_2 \cap L_\infty}$$

$$\dot{\hat{\theta}} = -\Gamma \nabla J_t(\hat{\theta})$$
$$= -\Gamma \bar{e} \Phi$$

$$|\dot{\hat{\theta}}| = |\Gamma \bar{e} \Phi|$$
$$\leq |\Gamma| \; |\bar{e}| \, |\Phi|$$

spectral norm $\qquad L_2 \cap L_\infty$

$$\Rightarrow \boxed{\dot{\hat{\theta}} \in L_2 \cap L_\infty}$$

(ii) convergence of $\hat{\theta}$ to θ and PE property

$$\tilde{\theta} = \hat{\theta} - \theta \qquad (\theta \in \mathbb{R}^{2n} \text{ — constant vector of sys. parameters})$$
$$\dot{\tilde{\theta}} = \dot{\hat{\theta}} = -\Gamma \bar{e} \Phi$$

where $\bar{e} = \Phi^T \tilde{\theta}$

Define $A(t) := -\Gamma \Phi(t) \Phi(t)^T \in \mathbb{R}^{2n \times 2n}$

$c(t) := \Phi(t) \in \mathbb{R}^{2n}$

Then we have the LTV system

$$\dot{\tilde{\theta}} = A(t) \tilde{\theta}$$
$$\bar{e} = c(t)^T \tilde{\theta}$$

with "state" $\tilde{\theta}$ and "output" $\bar{e}$.

We want to show $\tilde{\theta} \to 0$ exponentially fast.

This is implied by Uniform Complete Observability (UCO) property of $(A(t), c(t)^T)$.

Lemma (Ioannou - Sun) An LTV system $(A(t), C(t))$ is UCO if the pair $(A(t) + L(t)C(t), C(t))$ is UCO for some <u>bdd</u> output injection matrices $L(t)$.

We have $\dot{\tilde{\theta}} = A(t)\tilde{\theta}$ $\quad A(t) = -\Gamma\Phi(t)\Phi(t)^T$

$\bar{e} = c(t)^T\tilde{\theta}$ $\quad c(t) = \Phi(t)$

Take $L(t) = \Gamma\Phi(t)$

$$A(t) + L(t)c(t)^T = \underbrace{-\Gamma\Phi(t)\Phi(t)^T + \Gamma\Phi(t)\Phi(t)^T}_{=0}$$

Now we look at: $\dot{\tilde{\theta}} = 0$ $\quad (A(t) + L(t)c(t)^T)$

$\bar{e} = c(t)^T\tilde{\theta}$

For __this__ system, can write down observability Gramian easily:

$$M(t_0, t_0+T_0) = \int_{t_0}^{t_0+T_0} \Phi(t_0,t)^T c(t)c(t)^T \Phi(t_0,t)\,dt$$

where $\Phi(t_0, t) = I$ $\quad [\tilde{\theta}(t) = \tilde{\theta}(0)]$

$$\Rightarrow M(t_0, t_0+T_0) = \int_{t_0}^{t_0+T_0} \Phi(t)\Phi(t)^T dt \geq \alpha_0 T_0 I$$

(by PE assumption)

Thus, $\dot{\tilde{\theta}} = A(t)\tilde{\theta}$

$\bar{e} = c(t)^T\tilde{\theta}$

is UCO, so $\tilde{\theta}(t) \to 0$ exponentially fast. ∎