Adaptive Backstepping

<u>Review:</u> systematic methodology for constructing CLFs and stabilizing controllers for a wide class of nonlinear systems

$$\left.\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= u \end{aligned}\right\}$$

goal: CLF $V(x,\xi)$
stabilizing control law $u = k(x,\xi)$

Base system: $\dot{x} = f(x) + g(x)u$

- if this system admits a CLF $V_0(x)$ and a stab. controller $k_0(x)$ [assumed to be C^1], then we can construct:

CLF $V_1(x,\xi) = V_0(x) + \frac{1}{2}(\xi - k_0(x))^2$

stab. control law $u = k_1(x,\xi)$

The procedure can be iterated:

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{m-1} &= \xi_m \\ \dot{\xi}_m &= u \end{aligned}$$

$\begin{matrix} V_0(x) \\ k_0(x) \end{matrix} \longrightarrow \begin{matrix} V_1(x,\xi_1) \\ k_1(x,\xi_1) \end{matrix} \longrightarrow \cdots$

$\cdots \longrightarrow \begin{matrix} V_m(x,\xi_1,\dots,\xi_m) \\ u = k_m(x,\xi_1,\dots,\xi_m) \end{matrix}$

Need k_0 to be C^m.

Adaptive backstepping?

<u>Example</u>

$$\begin{aligned} \dot{x} &= \theta x + \xi \\ \dot{\xi} &= u \end{aligned}$$

$x, \xi \in \mathbb{R}$ (state variables)
$u \in \mathbb{R}$ (control)

Consider $\dot{x} = \theta x + u$ $\quad x, u \in \mathbb{R}$

Dynamic control: $u = -(\hat{\theta} + 1)x =: k_0(x, \hat{\theta})$
$\dot{\hat{\theta}} = x^2$

CLF: $V_0(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\hat{\theta} - \theta)^2$

Take this as base system: $\dot{x} = \theta x + u$
$\dot{\hat{\theta}} = x^2$

CLF $V_0(x, \hat{\theta})$, stab. control law $k_0(x, \hat{\theta}) = -(\hat{\theta}+1)x$

Apply backstepping to:

$$\begin{cases} \dot{x} = \theta x + \xi \\ \dot{\hat{\theta}} = x^2 \\ \dot{\xi} = u \end{cases}$$

$z := (x, \hat{\theta})^T$

$\dot{z} = f(z) + g(z)\xi$

where $f(z) = \begin{pmatrix} \theta x \\ x^2 \end{pmatrix}$, $g(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\longrightarrow \quad \begin{aligned} \dot{z} &= f(z) + g(z)\xi \\ \dot{\xi} &= u \end{aligned}$$

$$V_0(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}\underbrace{(\hat{\theta} - \theta)}_{:= \tilde{\theta}}{}^2$$

(parameter identification mismatch)

$$k_0(x, \hat{\theta}) = -(\hat{\theta}+1)x$$

Propose new CLF:

$$\begin{aligned} V_1(x, \hat{\theta}, \xi) &= V_0(x, \hat{\theta}) + \frac{1}{2}|\xi - k_0(x, \hat{\theta})|^2 \\ &= \frac{1}{2}x^2 + \frac{1}{2}\tilde{\theta}^2 + \frac{1}{2}(\xi + (\hat{\theta}+1)x)^2 \end{aligned}$$

Augmented system: $\dot{x} = \theta x + \xi$
$\dot{\hat{\theta}} = x^2$
$\dot{\xi} = u$

$$\dot{V}_1 = \frac{\partial V_1}{\partial x}\dot{x} + \frac{\partial V_1}{\partial \hat{\theta}}\dot{\hat{\theta}} + \frac{\partial V_1}{\partial \xi}\dot{\xi}$$

$$\frac{\partial V_1}{\partial x} = x + (\xi + (\hat{\theta}+1)x)(\hat{\theta}+1)$$

$$\frac{\partial V_1}{\partial \hat{\theta}} = \tilde{\theta} + (\xi + (\hat{\theta}+1)x)x$$

$$\frac{\partial V_1}{\partial \xi} = \xi + (\hat{\theta}+1)x$$

Reminder:
$\xi + (\hat{\theta}+1)x = \xi - k_0$

$$\dot{V}_1 = \Big[x + (\xi + (\hat\theta + 1)x)(\hat\theta + 1)\Big](\theta x + \xi)$$

$$+ \Big[\tilde\theta + (\xi + (\hat\theta + 1)x)x\Big]x^2 \qquad \Big[\dot{\hat\theta} = x^2\Big]$$

$$+ \Big[\xi + (\hat\theta + 1)x\Big]u$$

$$= x(\theta x + \xi) + \tilde\theta x^2$$

$$+ (\xi + (\hat\theta + 1)x)\Big[u + x^3 + (\hat\theta + 1)(\theta x + \xi)\Big]$$

$$= x(\theta x - \hat\theta x - x) + \tilde\theta x^2 + x(\xi + (\hat\theta + 1)x)$$

$$+ (\xi + (\hat\theta + 1)x)\Big[u + x^3 + (\hat\theta + 1)(\theta x + \xi)\Big]$$

$$= -x^2$$

$$+ (\xi + (\hat\theta + 1)x)\Big[u + x + x^3 + (\hat\theta + 1)(\theta x + \xi)\Big]$$

So far, so good?

$$\dot{x} = \theta x + u$$
$$\dot{\hat\theta} = x^2 \qquad u = -(\hat\theta + 1)x$$
$$\dot{V}_0(x, \hat\theta) = -x^2$$

Not so fast! The highlighted term depends on the unknown θ!

Here, we need to redesign (or adjust) $\dot{\hat\theta}$

$$\dot{x} = \theta x + \xi$$
$$\dot{\hat\theta} = \tau(x, \hat\theta, \xi)$$
$$\dot{\xi} = u \qquad u = k_1(x, \hat\theta, \xi)$$

$$\dot V_1(x,\hat\theta,\xi) = \frac{\partial V_1}{\partial x}\dot x + \frac{\partial V_1}{\partial \hat\theta}\dot{\hat\theta} + \frac{\partial V_1}{\partial \xi}\dot\xi$$

$$= \left[x + (\xi + (\hat\theta+1)x)(\hat\theta+1)\right](\theta x+\xi)$$
$$+ \left[\tilde\theta + (\xi + (\hat\theta+1)x)x\right]\tau \qquad \tau = \tau(x,\hat\theta,\xi)$$
$$+ (\xi + (\hat\theta+1)x)u$$

$$= x(\theta x+\xi) + \boxed{\tilde\theta\tau}$$
$$+ (\xi + (\hat\theta+1)x)\left[u + x\tau + (\hat\theta+1)(\underbrace{\theta}_{=\hat\theta-\tilde\theta} x+\xi)\right]$$

Plan: choose τ first
pick u (with τ chosen) to get $\dot V_1 < 0$

$$u + x\tau + (\hat\theta+1)(\theta x+\xi)$$
$$= u + x\tau + (\hat\theta+1)((\hat\theta-\tilde\theta)x+\xi)$$
$$= u + x\tau + (\hat\theta+1)(\hat\theta x+\xi) - \boxed{\tilde\theta(\hat\theta+1)x}$$

$$\dot V_1 = x(\theta x+\xi) + \tilde\theta\left(\tau - (\xi+(\hat\theta+1)x)(\hat\theta+1)x\right)$$
$$+ (\xi + (\hat\theta+1)x)\left[u + x\tau + (\hat\theta+1)(\hat\theta x+\xi)\right]$$

$$x(\theta x+\xi) = x\left(\theta x - (\hat\theta+1)x + \xi + (\hat\theta+1)x\right)$$
$$= x\left((\theta-\hat\theta)x - x\right) + (\xi + (\hat\theta+1)x)x$$
$$= -x^2 - \tilde\theta x^2 + (\xi + (\hat\theta+1)x)x$$

$$\dot V_1 = -x^2 + \tilde\theta\left(\tau - x^2 - (\xi+(\hat\theta+1)x)(\hat\theta+1)x\right)$$
$$+ (\xi + (\hat\theta+1)x)\left[u + x + x\tau + (\hat\theta+1)(\hat\theta x+\xi)\right]$$

$$\Rightarrow \text{take } \tau(x,\hat\theta,\xi) = x^2 + (\xi + (\hat\theta+1)x)(\hat\theta+1)x$$

$= 0$

already chosen

$$\dot V = -x^2 + (\xi + (\hat\theta + 1)x)[u + x + x\tau + (\hat\theta + 1)(\hat\theta x + \xi)]$$

$$u = -x - x\tau - (\hat\theta + 1)(\hat\theta x + \xi) - \underbrace{(\xi + (\hat\theta + 1)x)}_{\xi - k_0}$$

Then:

$$\dot V_1(x, \hat\theta, \xi) = -x^2 - (\xi + (\hat\theta + 1)x)^2$$

< 0 when $(x, \xi) \neq (0, 0)$

Upshot:

$$\left.\begin{aligned} \dot x &= \theta x + u \\ \dot{\hat\theta} &= \tau_0(x, \hat\theta) \\ u &= k_0(x, \hat\theta) \end{aligned}\right\} \text{ base system}$$

$$\begin{aligned} \dot x &= \theta x + \xi \\ \dot{\hat\theta} &= \tau_0(x, \hat\theta) + \tau_1(x, \hat\theta, \xi) \\ \dot\xi &= u \end{aligned}$$

$u = k_1(x, \hat\theta, \xi)$

$$V_1(x, \hat\theta, \xi) = V_0(x, \hat\theta) + \frac{1}{2}|\xi - k_0(x, \hat\theta)|^2$$

$$\tau_0 \longrightarrow \tau_0 + \tau_1$$

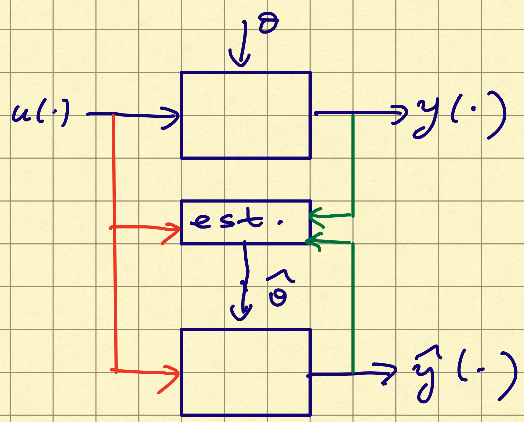
$$\begin{aligned} \dot x &= \theta x + \xi_1 \\ \dot\xi_1 &= \xi_2 \\ \dot\xi_2 &= u \end{aligned}$$

$\tau_0 + \tau_1 \longrightarrow \tau_0 + \tau_1 + \tau_2$

$\tau_0, \tau_1, \tau_2 \cdots$ - tuning fcns

Preview for next lecture: parameter estimation

System identification:



$\tilde{y}(t) := \hat{y}(t) - y(t)$

$\tilde{y}(t)$ depends on $\hat{\theta}$

$e_t(\hat{\theta}) := \frac{1}{2}(\hat{y}(t) - y(t))^2$

— a fcn of $\hat{\theta}$

$\hat{\theta}(t) \leftarrow \min e_t(\hat{\theta})$ over $\hat{\theta}$

Under some conditions on the plant and $u(\cdot)$,
can guarantee $\hat{\theta}(t) \to \theta$ as $t \to \infty$
[θ can be identified].