Lyapunov-Based design

Back to example: $\dot{x} = \theta x + u \qquad (\theta \in \mathbb{R})$

Reasoning:

1) Certainty Equivalence Principle
(take $\hat{\theta}$, act as if it is true θ)

$$u = -(\hat{\theta}+1)x$$

Heuristic: $\hat{\theta}$ may not converge to θ.

2) Choice of a tuning law:

$$u = -(\hat{\theta}+1)x, \qquad \dot{\hat{\theta}} = x^2$$

- dominate lin. growth of $\dot{x}$ with quadratic growth of $\dot{\hat{\theta}}$

3) Choice of candidate Lyapunov fcn

$$\dot{x} = (\theta - \hat{\theta} - 1)x$$
$$\dot{\hat{\theta}} = x^2$$

$$V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\hat{\theta} - \theta)^2$$

- heuristic b/c $\hat{\theta}$ may not converge to θ.

Is a <u>more principled</u> approach available?

Yes: Lyapunov-based design

Start by choosing a good V (candidate LF)

$$\dot{x} = \theta x + u$$
$$\dot{\hat{\theta}} = f(\hat{\theta}, x) \qquad u = k(\hat{\theta}, x)$$
$$V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\hat{\theta} - \theta)^2$$

$$\dot{\hat\theta} = f(\hat\theta, x)$$
$$u = k(\hat\theta, x)$$

$$\dot V = x\dot x + (\hat\theta - \theta)\dot{\hat\theta}$$
$$= x(\theta x + \underbrace{k(\hat\theta, x)}_{=u}) + (\hat\theta - \theta) f(\hat\theta, x)$$
$$= \theta x^2 - \theta f(\hat\theta, x) + x k(\hat\theta, x) + \hat\theta f(\hat\theta, x)$$
$$= \theta(x^2 - f(\hat\theta, x)) + x k(\hat\theta, x) + \hat\theta f(\hat\theta, x)$$

$$\Rightarrow \boxed{f(\hat\theta, x) = x^2}$$

[arrange to have $\dot V$ is indep. of θ]

Now we have $\dot x = \theta x + k(\hat\theta, x)$

$\dot{\hat\theta} = x^2$

$$\dot V = x k(\hat\theta, x) + \hat\theta x^2$$

Choose $k(\cdot,\cdot)$ to give $\dot V = -x^2$

$$x k(\hat\theta, x) + \hat\theta x^2 = -x^2$$
$$x k(\hat\theta, x) = -(\hat\theta + 1)x^2$$

if $x \neq 0$, then $\boxed{k(\hat\theta, x) = -(\hat\theta + 1)x}$

Summary: $\dot x = \theta x + k(\hat\theta, x)$

$\dot{\hat\theta} = f(\hat\theta, x)$

– need to design $k(\cdot,\cdot)$, $f(\cdot,\cdot)$

$$V(x, \hat\theta) = \tfrac{1}{2}x^2 + \tfrac{1}{2}(\hat\theta - \theta)^2$$
$$\dot V = (\theta x^2 + x k(\hat\theta, x)) + (\hat\theta - \theta) f(\hat\theta, x)$$

– choose $f(\cdot,\cdot)$, $k(\cdot,\cdot)$ to give $\dot V = -x^2$

Same thing if we want $\dot V = -cx^2$ $(c > 0)$

$u = -(\hat\theta + c)x$

$\dot{\hat\theta} = x^2$

General Framework: Control Lyapunov Functions

$\dot{x} = f(x,u) \qquad x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m$

Stabilize to equilibrium: assume $f(0,0)=0$

$V : \mathbb{R}^n \to [0,\infty)$ candidate LF (C^1, p.d.)

Choose $u = k(x)$ [static state feedback]

s.t. $\dot{V} = \underbrace{\frac{\partial V}{\partial x}}_{\text{Jacobian} \in \mathbb{R}^{1\times n}} \cdot f(x, k(x)) < 0 \qquad \forall x \neq 0$

__Def.__: a candidate LF V is a Control Lyapunov Function (CLF) for $\dot{x} = f(x,u)$ if

$\forall x \neq 0 \quad \exists u \in \mathbb{R}^m$ s.t. $\frac{\partial V}{\partial x} f(x,u) < 0$.

Equivalently, V is a CLF if

$$\boxed{\inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x} f(x,u) \right\} < 0} \qquad \forall x \neq 0.$$

__Problem__: find a continuous state feedback law $u = k(x)$ s.t.

1) $\dot{x} = f(x, k(x))$ [closed-loop] is A.S.
2) $\frac{\partial V}{\partial x} f(x, k(x)) < 0$ for all $x \neq 0$.

[also want $k(0)=0$].

__Caution__: existence of CLF does not imply existence of cont. $k(\cdot)$!

$\forall x \neq 0$: $S(x) := \left\{ u \in \mathbb{R}^m : \underbrace{\frac{\partial V}{\partial x} f(x,u)}_{\text{?}} < 0 \right\}$

Want a selection rule $x \mapsto k(x)$ s.t. $k(x) \in S(x)$ and $x \mapsto k(x)$ is cont.

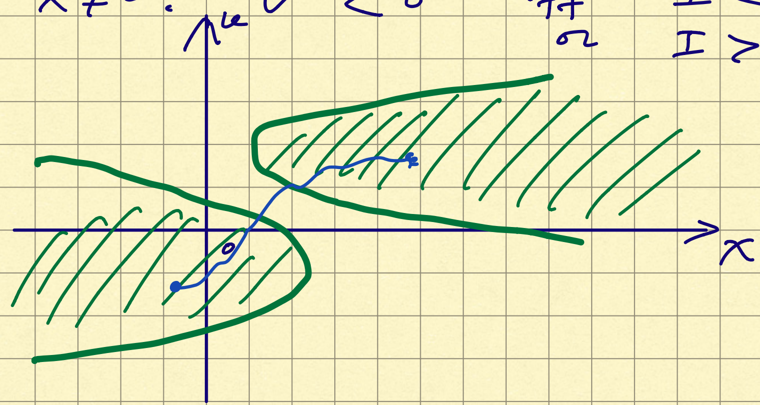
Counterex.: (E. Sontag, H. Sussmann)

$$\dot{x} = x\left[(u-1)^2 - (x-1)^2\right]\left[(u+1)^2 + (x-2)\right]$$

$$V(x) = \tfrac{1}{2}x^2$$

$$\dot{V}(x,u) = x^2 \overbrace{\left[(u-1)^2 - (x-1)^2\right]}^{\mathrm{I}}\overbrace{\left[(u+1)^2 + (x-2)\right]}^{\mathrm{II}}$$

$x \neq 0$: $\dot{V} < 0$ iff $\mathrm{I} < 0$ & $\mathrm{II} > 0$
or $\mathrm{I} > 0$ & $\mathrm{II} < 0$



– no continuous selection of x from $S(x)$

However, the continuous control sel. problem always has a solution for control-affine systems:

$$\dot{x} = f(x) + G(x)u \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

$$= f(x) + \sum_{i=1}^{m} u_i g_i(x) \qquad f: \mathbb{R}^n \to \mathbb{R}^n$$

$$G: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$$

$$G(x) = \left[\, g_1(x) \;\vdots\; g_2(x) \;\vdots\; \cdots \;\vdots\; g_m(x) \,\right]$$

$u \mapsto f(x) + G(x)u$ is affine in u for each x

1) Continuous stabilizing feedback exists (Z. Artstein, 1983)

2) Stabilizing feedback can be given in closed form (Sontag, 1989)

Sontag's Universal Formula

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \qquad \dot{V} = \frac{\partial V}{\partial x}\left(f(x) + G(x)u\right)$$

CLF: $\displaystyle\inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x} f(x) + \sum_{i=1}^{m} u_i \frac{\partial V}{\partial x} g_i(x) \right\} < 0$

for $x \neq 0$

Claim: this is equivalent to asking that

$$\frac{\partial V}{\partial x} g_i(x) = 0, \ \forall i \implies \frac{\partial V}{\partial x} f(x) < 0 .$$

Proof: two cases:

- $\frac{\partial V}{\partial x} g_i(x) \neq 0$ for at least one $i \in \{1, \dots, m\}$
  - choose u_i to give $\dot{V} < 0$
- $\frac{\partial V}{\partial x} g_i(x) = 0 \quad \forall i$ [at this x, we lose control authority]

$$\dot{V} < 0 \iff \frac{\partial V}{\partial x} f(x) < 0 .$$

$\blacksquare$

Notation:

$$a(x) := \frac{\partial V}{\partial x} f(x) \qquad \mathbb{R}^n \to \mathbb{R}$$

$$b(x) := \left(\frac{\partial V}{\partial x} g_1(x), \dots, \frac{\partial V}{\partial x} g_m(x) \right)^T \qquad \mathbb{R}^n \to \mathbb{R}^m$$

CLF: $\|b(x)\| = 0 \implies a(x) < 0$

Sontag's Universal Controller:

$u = k(x) = K(a(x), b(x))$ where

$$K(a, b) := \begin{cases} -\dfrac{a + \sqrt{a^2 + \|b\|^4}}{\|b\|^2}\, b & , b \neq 0 \\ 0 & , b = 0 \end{cases}$$

[motivation: LQR, Riccati equ., next lec.]

$x \mapsto k(x)$ is continuous

For $x \neq 0$:

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \sum_{i=1}^{m} k_i(x) \frac{\partial V}{\partial x} g_i(x)$$

$$= a(x) + \sum_{i=1}^{m} k_i(x) b_i(x)$$

$$= a(x) - \sum_{i=1}^{m} \frac{a(x) + \sqrt{a^2(x) + \|b(x)\|^4}}{\|b(x)\|^2} |b_i(x)|^2 \quad [\|b(x)\| \neq 0]$$

$$= a(x) - \left(a(x) + \sqrt{a^2(x) + \|b(x)\|^4}\right) \underbrace{\sum_{i=1}^{m} \frac{|b_i(x)|^2}{\|b(x)\|^2}}_{=1}$$

$$= -\sqrt{a^2(x) + \|b(x)\|^4} < 0$$