Universal Regulation

Review: $\dot{y} = ay + bu$ ($a \in \mathbb{R}$, $b \neq 0$: unknown)

Nussbaum (1983), simplified by Willems-Byrnes:

$\dot{k} = y^2$, $\quad u = -N(k)\,ky$

$N : \mathbb{R} \to \mathbb{R}$ (Nussbaum gain)

For $S(k) := \int_0^k \sigma N(\sigma)\,d\sigma$, require:

$$\left.\begin{aligned} &\inf_{k \geq 1} \frac{1}{k} S(k) = -\infty \\ &\sup_{k \geq 1} \frac{1}{k} S(k) = +\infty \end{aligned}\right\}$$

exploration (switch from +1 to −1 infinitely often)
exploitation (maintain const. sgn for nontrivial periods of time)

Ex.:

$$N(k) = \begin{cases} +1, & n^2 \leq |k| < (n+1)^2, \quad n = 0, 2, \ldots \\ -1, & n^2 \leq |k| < (n+1)^2, \quad n = 1, 3, \ldots \end{cases}$$

Closed-loop system:

$$\begin{aligned} \dot{y} &= (a - bkN(k))y \\ \dot{k} &= y^2 \end{aligned}$$

Then universal regulation is achieved.

Universal Regulation for Higher-Order plants

SISO $\quad u, y$ scalar
$\quad x \in \mathbb{R}^n$, $\; n > 1$

Consider the following class of LTI SISO sys:

$x = \begin{pmatrix} y \\ z \end{pmatrix}$ $\quad y \in \mathbb{R}$ (output)
$\quad z \in \mathbb{R}^{n-1}$

$$\boxed{\begin{aligned} \dot{y} &= ay + bu + \underbrace{c^T z}_{\text{new term}} \\ \dot{z} &= Az + dy \end{aligned}}$$

Parameters: $a \in \mathbb{R},\ b \neq 0$
$c \in \mathbb{R}^{n-1}$
$A \in \mathbb{R}^{(n-1)\times(n-1)}$
$d \in \mathbb{R}^{n-1}$

Goal: universal regulation

$$x(t) := \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} \longrightarrow 0 \quad \text{as } t \to \infty$$

all signals in closed-loop sys. bdd

control has to only use output feedback

$$\begin{aligned} \dot{k} &= f(k, y) \\ u &= h(k, y) \end{aligned}$$

Motivation: $\dot{y} = ay + bu + c^T z$

if a, b, c known, can choose u to give $\dot{y} \equiv 0$

$y(t) \equiv 0$ [assuming $y(0) = 0$]

$$u = -\frac{1}{b}(ay + c^T z)$$

$$\dot{y} = ay + b \cdot \left(-\frac{1}{b}(ay + c^T z)\right) + c^T z$$

$$= \cancel{ay} - \cancel{ay} - \cancel{c^T z} + \cancel{c^T z} = 0$$

$$\dot{z} = Az + d\underbrace{y}_{\equiv 0} \longrightarrow \dot{z} = Az$$

$\Rightarrow$ A has to be Hurwitz (all eig(A) have negative real parts)

Class of plants: $\mathcal{C} = \left\{ \begin{aligned} \dot{y} &= ay + bu + c^T z \\ \dot{z} &= Az + dy \end{aligned} \right.,$

where $a \in \mathbb{R},\ b \neq 0,\ c, d \in \mathbb{R}^{n-1},\ A \in \mathbb{R}^{(n-1)\times(n-1)}$ Hurwitz $\Big\}$

Remark: plants in $\mathcal{C}$ are in **normal form**

$$x = \begin{pmatrix} v \\ z \end{pmatrix}$$

$$\begin{aligned}
\dot v_1 &= v_2 \\
\dot v_2 &= v_3 \\
&\vdots \\
\dot v_r &= a(v,z) + \underbrace{b(v,z)}_{\neq 0} u \\
\dot z &= q(v, z)
\end{aligned}$$

r: relative degree

can choose u to make $\underbrace{v_1 = \dots = v_r = 0}$ (i.c.!)

$$u(v,z) := -\frac{a(v,z)}{b(v,z)}$$

$$\dot z = q(0,z) \quad - \quad \text{zero dynamics}$$

zero dynamics are stable [minimum-phase property]

Class $\mathcal{C}$ consists of: SISO LTI systems that have rel. deg. 1 and the min. phase property.

Claim: Nussbaum gain still works!

Proof:

$$\begin{aligned}
\dot y &= (a - bN(k)k)y + c^T z \\
\dot z &= Az + dy \\
\dot k &= y^2
\end{aligned}$$

goal: $y(t), z(t) \to 0$
all signals remain bdd

$$V(y) = \frac{1}{2}y^2$$

$$\dot{V} = y\dot{y} = y\overbrace{\left((a - bN(k)k)y + c^T z\right)}^{\dot{y}}$$

$$= (a - bN(k)k)y^2 + yc^T z \qquad (\dot{k} = y^2)$$

$$= (a - bN(k)k)\dot{k} + \underbrace{yc^T z}_{\text{new term}}$$

Integrate:

$$\frac{1}{2}y^2(t) = \int_0^t (a - bN(k(s))k(s))\dot{k}(s)ds + \int_0^t y(s)\underbrace{c^T z(s)}_{\text{"output" of } \dot{z} = Az + dy}ds + C_1$$

$$= \int_0^t a\dot{k}(s)ds - b\int_0^t N(k(s))k(s)\dot{k}(s)ds$$

$$= ak(t) - b\mathcal{S}(k(t)) + C_2$$

$= ?$ - need some facts about exp-stable LTI systems

Let $\dot{z} = Fz + Gv$ be an exp. stable system (F is Hurwitz) with output $c^T z$

Fact 1: $\exists\, c_1, c_2 > 0$ s.t.

$$\int_0^t (c^T z(s))^2 ds \le c_1|z(0)|^2 + c_2\int_0^t y^2(s)ds, \quad \forall t \ge 0$$

$$= \int_0^t y(s)c^T z(s)ds \qquad \begin{array}{l}\dot{z} = Az + dy \\ c^T z \text{ "output"}\end{array}$$

$$\le \sqrt{\int_0^t y^2(s)ds}\cdot\sqrt{\int_0^t (c^T z(s))^2 ds} \qquad \text{(Cauchy-Schwarz)}$$

$$\le \underbrace{\frac{1}{2}\int_0^t y^2(s)ds}_{1} + \underbrace{\frac{1}{2}\int_0^t (c^T z(s))^2 ds}_{2} \qquad ab \le \frac{a^2}{2} + \frac{b^2}{2}$$

$\leftarrow$ Fact 1

$$\le c_3|z(0)|^2 + c_4\int_0^t y^2(s)ds \qquad [c_3, c_4 > 0]$$

Recall: $\quad y^2 = \dot{k}$

$$\leq c_3 |z(0)|^2 + c_4 (k(t) - k(0))$$

$$\therefore \boxed{\frac{y^2(t)}{2} \leq (a + c_4) k(t) - b\, \mathcal{S}(k(t)) + \bar{C}}$$

$\forall t \geq 0$

LHS ≥ 0, so $k(t)$ has to remain bdd

$\Rightarrow$ $y(t)$ has to remain bdd

$$\int_0^t y^2(s)\, ds = k(t) - k(0) \leq \bar{\bar{C}} \quad [k(t) \text{ is bdd}]$$

for all t

$\Rightarrow$ $y \in L_2$: $\displaystyle\lim_{t\to\infty} \int_0^t y^2(s)\, ds < \infty$

$$\begin{aligned} \dot{y} &= ay + bu + c^T z \\ &= (a - bN(k)k)\, y + c^T z \end{aligned}$$

Need z to be bdd!

Fact 2: If $\dot{z} = Fz + Gv$ is exp. stable and v is bdd, then z is bdd

y bdd $\Rightarrow$ z is bdd (A is Hurwitz)

$\left.\begin{array}{l} y, \dot{y} \text{ bdd} \\ y \in L_2 \end{array}\right\} \Rightarrow$ $y(t) \to 0$ as $t \to \infty$ by Barbalat

Still need to show $z \to 0$!

Fact 3: If $\dot{z} = Fz + Gv$ is exp stable, then

$v \in L_2$ or $v \to 0$ $\Rightarrow$ $z \to 0$

Apply to $\dot{z} = Az + dy$ $\quad y \in L_2,\ y \to 0 \Rightarrow z \to 0.$

<u>Upshot:</u> can achieve universal regulation for SISO higher-order plants that are controllable, have rel. deg. 1, and are min. phase

$$\mathcal{C} = \left\{ \begin{array}{l} \dot{y} = ay + bu + c^T z \\ \dot{z} = Az + dy \end{array} \; : \; \begin{array}{l} a \in \mathbb{R}, \; b \neq 0 \\ c, d \in \mathbb{R}^{n-1} \\ A \in \mathbb{R}^{(n-1)\times(n-1)} \text{ Hurwitz} \end{array} \right\}.$$

- Nussbaum's controller does not have finite gain (for an apriori value):

$$\dot{k} = y^2, \qquad u = -N(k)ky$$

because $\mathcal{C}$ contains plants that are arbitrarily close to unstable ($a > 0$) and uncontrollable ($b \to 0$) plants.