# Universal Regulators for Linear Systems

$\dot{y} = ay + bu \qquad u(t), y(t) \in \mathbb{R}$

$a \in \mathbb{R}, \quad b \neq 0 \qquad$ (both unknown)

Goal: universal regulation $\begin{cases} y(t) \to 0 \text{ as } t \to \infty \\ \text{all signals bdd} \end{cases}$

for arbitrary initial conditions $y(0)$, any other i.c. for the controller.

From Lec. 3: sign$(b)$ known $\Rightarrow$ universal reg. possible

Take $b > 0$ ($b < 0$ case is similar)

Dynamic controller: $\dot{k} = y^2, \quad u = -ky$

Showed $k(t), y(t)$ bdd $\Rightarrow$ $\dot{y}(t)$ bdd

$y \in L_2$: $\int_0^\infty y^2(t)\,dt < \infty \Rightarrow y(t) \to 0$ as $t \to \infty$ by Barbalat

Now, assume sign$(b)$ unknown (but $b \neq 0$)

Some fundamental limits:

take controllers of the form $\begin{aligned} \dot{z} &= f(z, y) \\ u &= h(z, y) \end{aligned}$

— our controller from before:

$\begin{aligned} \dot{z} &= y^2 \\ u &= -zy \end{aligned} \qquad \begin{aligned} f(z, y) &= y^2 \\ h(z, y) &= -zy \end{aligned}$

Consider a class of controllers w/ continuous rational dynamics:

$f(z, y)$ and $h(z, y)$ are ratios of polynomials in $z$ and $y$, denominators have no real roots

E.g. $f(z, y) = \dfrac{q(z, y)}{p(z, y)} \qquad q, p$ polynomials

$q(z, y) \neq 0$ for all real $z, y \in \mathbb{R}$

Claim : no controller of this form can achieve universal regulation! (when sign(b) is unknown)

[proved by R. Nussbaum in 1983]

Context: Morse (in 1983) conjectured that universal regulation is impossible if sign(b) is unknown

## Proof of the claim

① Controller must have nontrivial dynamics (in other words, $f$ cannot be $\equiv 0$)

Assume $z$ stays constant ($f \equiv 0$). Then the controller is static: $u = h(y)$.

Two cases:

$h(y) \neq 0$, $\forall y \in \mathbb{R}$ (e.g. $h(y) > 0$)

Then take $a = b = 1$, $y_0 > 0$

$\dot{y} = y + h(y)$
$y(0) = y_0 > 0$

$\Rightarrow$ $y(t)$ will keep increasing

$y \not\to 0$

$h(y_0) = 0$ for some $y_0$

Take $a = 0$, $b = 1$

$\dot{y} = h(y)$

$y(0) = y_0 > 0$

$y_0$ is an equilibrium, so $y(t) \equiv y_0 > 0$, $\forall t$

$y \not\to 0$

... the controller needs a state!

Dynamic controllers: $\dot{z} = f(z, y)$
$u = h(z, y)$

$f, h$ are cont. rational fcns, $f(z, y)$ is not identically $0$.

$\Rightarrow$ $\exists z_0$ s.t. $y \mapsto f(z_0, y) \not\equiv 0$

$$f(z_0, y) = \frac{q(z_0, y)}{p(z_0, y)}$$

$q(z_0, y)$ is a poly in $y$
$p(z_0, y)$ is a poly in $y$, $\neq 0$ for all $y$

$q(z_0,y)$ has finitely many roots

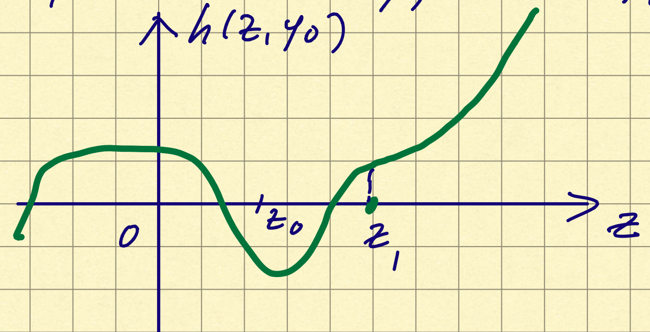
$\Rightarrow \exists\, y_0 > 0$ s.t. $\dfrac{q(z_0,y)}{p(z_0,y)} \neq 0$ for all $y \geq y_0$

(assume w.l.o.g. that $f(z_0,y) > 0$ for all $y \geq y_0 > 0$).

Now, $z \mapsto h(z,y_0)$ [cont., rat. fcn of $z$]

$\Rightarrow \exists\, z_1 \geq z_0$ s.t. $h(z,y_0) > 0$ for all $z \geq z_1$

by continuity, $h(z,y_0)$ is bdd from below for all $z \geq z_0$.

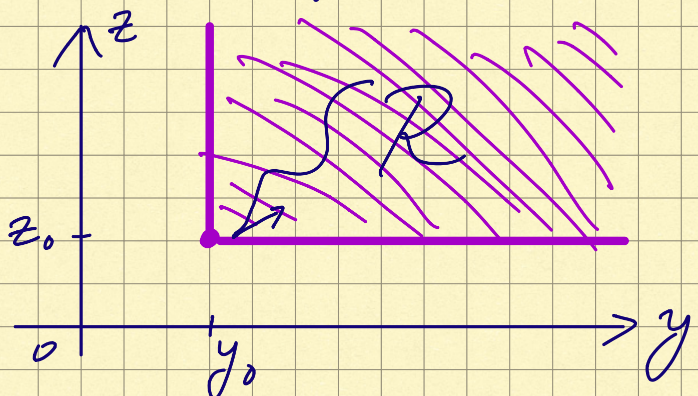



$$\boxed{h(z,y_0) \geq c \quad (c \in \mathbb{R}) \text{ for all } z \geq z_0}$$

Closed-loop dynamics:
$$\begin{cases} \dot{y} = ay + bh(z,y) \\ \dot{z} = f(z,y) \end{cases}$$

$ay_0 + bh(z,y_0)$ : $\exists\, b > 0$ s.t. $y_0 + bh(z,y_0) > 0$ for all $z \geq z_0$

$\mathcal{R} := \{(y,z) \in \mathbb{R}^2 : y \geq y_0,\ z \geq z_0\}$




Claim: if $(y(0), z(0)) \in \mathcal{R}$, then $(y(t), z(t)) \in \mathcal{R}$ for all $t \geq 0$

– $\mathcal{R}$ is a positively invariant set, and in particular $y(t) \not\to 0$ as $t \to \infty$!

**But:** universal regulation is achievable even with unknown sign(b), but controller must be able to switch the sign of feedback gain.

(Nussbaum, 1983)

**Main insight:** controller gain should flip back and forth between $>0$ and $<0$ infinitely often – explore

and maintain the same sign long enough to achieve stabilization – exploit

Nussbaum's controller (Willems - Byrnes, 1984)

$$\dot{y} = ay + bu \qquad a \in \mathbb{R}, \quad b \neq 0 \quad \text{unknown}$$

$$\dot{k} = y^2$$

$$u = -N(k)ky \qquad N : \mathbb{R} \to \mathbb{R} \quad \text{(Nussbaum gain)}$$

Require the following:

$$\mathcal{S}(k) := \int_0^k \sigma N(\sigma)\, d\sigma$$

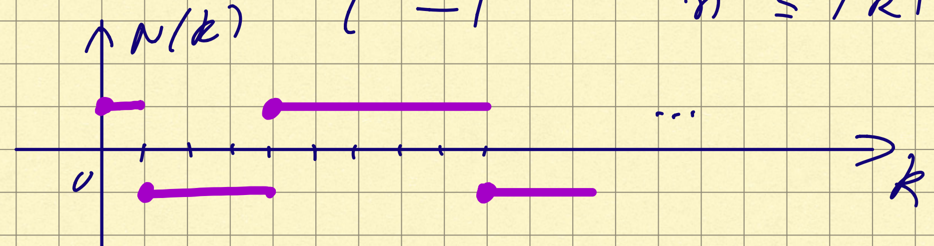
is s.t.

$$\sup_{k \geq 1} \frac{1}{k}\mathcal{S}(k) = +\infty$$

$$\inf_{k \geq 1} \frac{1}{k}\mathcal{S}(k) = -\infty$$

**Examples:** $N(k) = \sin\sqrt{|k|}$ (Willems-Byrnes)

$$N(k) = \begin{cases} +1, & n^2 \leq |k| \leq (n+1)^2, \quad n = 0, 2, \ldots \\ -1 & n^2 \leq |k| \leq (n+1)^2, \quad n = 1, 3, \ldots \end{cases}$$




(Willems-Byrnes)

**Proof** (Nussbaum, via Willems-Byrnes)

$$\dot{y} = ay - bN(k)ky$$
$$\dot{k} = y^2$$

$V(y) = \frac{y^2}{2}$ $\qquad \dot{V} = y(t)\dot{y}(t)$

$$\dot{V} = (a - bN(k)k)y^2$$
$$= (\underline{a} - bN(k)k)\underline{\dot{k}}$$

Integrate:

$$\frac{y^2(t)}{2} - \frac{y^2(0)}{2} = \underbrace{\int_0^t a\dot{k}(s)ds}_{1} - \overbrace{b\int_0^t N(k(s))k(s)\frac{\dot{k}(s)}{ds}}^{2}$$

$\sigma := k(s)$

$$= \underbrace{ak(t) - ak(0)}_{1} - \underbrace{b\int_{k(0)}^{k(t)} \sigma N(\sigma)d\sigma}$$

$$\int_{k(0)}^{k(t)} \sigma N(\sigma)d\sigma = \underbrace{\int_0^{k(t)} \sigma N(\sigma)d\sigma}_{S(k(t))} + \underbrace{\int_{k(0)}^{0} \sigma N(\sigma)d\sigma}_{\text{dep. only on } k(0)}$$

2

$$\Rightarrow \boxed{\frac{y^2(t)}{2} = ak(t) - bS(k(t)) + C}$$

LHS $\geq 0 \Rightarrow ak(t) - bS(k(t)) \geq -C, \; \forall t$

$\dot{k} = y^2 \Rightarrow k(t)$ is nondecreasing

$k(t)$ either grows w/o bound or reaches a finite limit

Recall: $\inf\limits_{k \geq 1} \frac{1}{k}S(k) = -\infty, \quad \sup\limits_{k \geq 1} \frac{1}{k}S(k) = +\infty$

If $k(t)$ grows w/o bound, then eventually it will reach some value $\bar{k}$ s.t.

$$a\bar{k} - b\,S(\bar{k}) < -c$$

[need to consider $b \lessgtr 0$]

Now, follow usual steps: $k(t)$ bdd $\Rightarrow y(t)$ bdd $\Rightarrow \dot{y}(t)$ bdd

$$\int_0^t y^2(s)\,ds = k(t) - k(0) \leq C'$$

$y \in L_2 \implies y(t) \to 0$ as $t \to \infty$ by Barbalat.

$$\downarrow$$

$\underbrace{\int_0^\infty y^2(t)\,dt}_{\|y\|_{L_2}}$ exists and is finite