Review: Weak Lyapunov Functions

$\dot{x} = f(x)$ $\qquad$ f cont., $\quad f(0)=0$

$V(x)$: candidate Lyapunov fcn

C^1, $\quad V(x) \geq 0$

<u>Thm</u> (weak Lyapunov criterion)

Suppose that $\dot{V}(x) = \nabla V(x)^T f(x)$ satisfies

$$\dot{V}(x) \leq -W(x), \qquad \forall x$$

for some cont. fcn W taking nonnegative values. Then, for every bounded trajectory $x(t)$,

$$W(x(t)) \longrightarrow 0 \quad \text{as} \quad t \to \infty.$$

<u>Reminder:</u> along the way, we used (and proved) Barbalat's lemma:

If $x(t)$, $\dot{x}(t)$ remain bounded and

$$\int_0^\infty W(x(t))\,dt$$

exists and is finite, then $W(x(t)) \longrightarrow 0$ as $t \to \infty$.

Connection to Observability

LTI system: $\quad \dot{x} = Ax \qquad x(t) \in \mathbb{R}^n$, $\quad A \in \mathbb{R}^{n \times n}$

$V(x) = x^T P x$ $\quad$ – candidate LF

$P = P^T > 0 \qquad v^T P v > 0 \quad \forall v$

$$\dot{V}(x) = x^T(PA + A^T P)x$$

$$\left[\dot{V}(x(t)) = \frac{d}{dt} V(x(t)) = \frac{d}{dt}\left\{x(t)^T P x(t)\right\}\right]$$

<u>Assume</u> $\exists$ a matrix C s.t. $\quad PA + A^T P \preceq -C^T C$:

$$\boxed{\underbrace{x^T(PA + A^T P)x}_{\dot{V}(x)} \leq -\underbrace{x^T C^T C x}_{(Cx)^T(Cx)}}$$

$C \in \mathbb{R}^{p \times n}$ for some p

Consider a fictitious output $y = Cx$

$$\dot{x} = Ax$$
$$y = Cx$$

If (A, C) is <u>an observable pair</u>, then

$y(t) \to 0$ as $t \to \infty$ $\Rightarrow$ $x(t) \to 0$ as $t \to \infty$.

Back to Thm on weak Lyapunov fcns:

$W(x) = x^T C^T C x$

$\dot{V}(x) \leq -W(x)$

$x(t)$ remains bdd (since $V(x)$ is p.d. and radially unbounded), so

$W(x(t)) = \| Cx(t) \|^2 \to 0$ as $t \to \infty$

$\Rightarrow$ $y(t) \to 0$ as $t \to \infty$

$\Rightarrow$ $x(t) \to 0$ as $t \to \infty$ by observability.

This can be extended to LTV (linear, time-varying systems) using Uniform Complete Observability [Kalman, 1960].

<u>Back to our example:</u>

$\dot{x} = \theta x + u$ $\quad$ ($\theta \in \mathbb{R}$ unknown)

goal: $x(t) \to 0$ as $t \to \infty$
$u(t)$ remains bdd $\forall t \geq 0$

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} (\theta - \hat{\theta} - 1)x \\ x^2 \end{pmatrix}$$

tuning law: $\dot{\hat{\theta}} = x^2$
$u = -(\hat{\theta} + 1)x$

$$V(x, \hat{\theta}) = \frac{x^2}{2} + \frac{(\hat{\theta} - \theta)^2}{2}$$

Using weak Lyapunov fcns, we showed:

$x(t) \to 0$ as $t \to \infty$

and $\hat{\theta}(t)$ remains bdd $\quad$ [$\Rightarrow$ $u(t)$ remains bdd]

Important: cannot guarantee $\hat{\theta}(t) \to \theta$ as $t \to \infty$.

Recall our first attempt at analysis:

$$V(x) = \frac{1}{2}x^2$$

$$\dot{V}(x) = x\dot{x}$$

$$= (\theta - \hat{\theta} - 1)x^2$$

$$\boxed{\begin{pmatrix} \dot{x} \\ \dot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} (\theta - \hat{\theta} - 1)x \\ x^2 \end{pmatrix}}$$

We will now show that adaptive regulation can be proved even for this choice of V!

$$\dot{V}(x) = (\theta - \hat{\theta} - 1)\,\dot{\hat{\theta}} \qquad \text{since } \dot{\hat{\theta}} = x^2$$

$$\dot{V}(x(t)) = (\theta - \hat{\theta}(t) - 1)\,\dot{\hat{\theta}}(t)$$

Integrate: from 0 to t

LHS: $\displaystyle\int_0^t \dot{V}(x(s))\,ds = \int_0^t \frac{d}{ds} V(x(s))\,ds = V(x(t)) - V(x(0))$

RHS:

$$\int_0^t (\theta - \hat{\theta}(s) - 1)\,\dot{\hat{\theta}}(s)\,ds$$

$$= \underbrace{\int_0^t (\theta - 1)\,\dot{\hat{\theta}}(s)\,ds}_{1} - \underbrace{\int_0^t \hat{\theta}(s)\,\dot{\hat{\theta}}(s)\,ds}_{2}$$

$1 = (\theta - 1)(\hat{\theta}(t) - \hat{\theta}(0)) = (\theta - 1)\hat{\theta}(t) + \underbrace{C_1}_{\text{depends only on } \hat{\theta}(0)}$

$2 = \frac{1}{2}\hat{\theta}^2(s)\Big|_0^t = \frac{1}{2}\hat{\theta}^2(t) + \underbrace{C_2}_{\text{depends only on } \hat{\theta}(0)}$

$$V(x(t)) = (\theta-1)\hat{\theta}(t) - \frac{1}{2}\hat{\theta}^2(t) + C$$

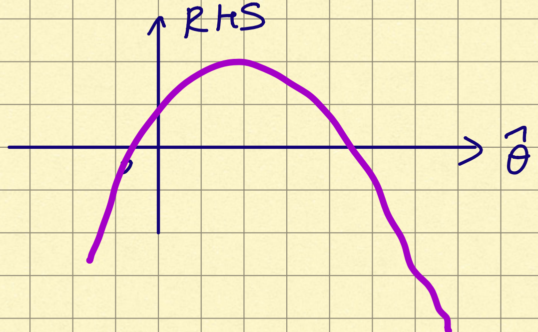
depends only on $x(0)$, $\hat{\theta}(0)$

$$\boxed{\frac{x^2(t)}{2} = (\theta-1)\hat{\theta}(t) - \frac{1}{2}\hat{\theta}^2(t) + C}$$

Some observations:

1) LHS ≥ 0 $\Rightarrow$ RHS ≥ 0 for all $t \geq 0$

2) RHS (as a function of $\hat{\theta}(t)$) is a concave quadratic fcn



- if $\hat{\theta}(t)$ increases w/o bound, then eventually RHS < 0

Thus: $\hat{\theta}(t)$ remains bdd $\forall t \geq 0$

In detail: $\dot{\hat{\theta}}(t) = x^2(t) \Rightarrow$ $\hat{\theta}(t)$ is monotone nondecreasing

$\Rightarrow$ so, $\hat{\theta}(t)$ either grows w/o bound or remains bdd

But if $\hat{\theta}(t)$ grows w/o bound, then eventually RHS will be negative, which cannot happen.

$$x^2(t) = 2\left\{(\theta-1)\hat{\theta}(t) - \frac{1}{2}\hat{\theta}^2(t) + C\right\}$$

$$\sup_{t\geq 0} x^2(t) = 2\sup_{t\geq 0}\left\{(\theta-1)\hat{\theta}(t) - \frac{1}{2}\hat{\theta}^2(t) + C\right\} < \infty$$

$\hat{\theta}(t)$ remains bdd, $x(t)$ remains bdd

$|\dot{x}(t)| = |(\theta - \hat{\theta}(t) - 1)x(t)|$ remains bdd

$$0 \leq \int_0^t x^2(s)\,ds = \int_0^t \dot{\hat{\theta}}(s)\,ds = \hat{\theta}(t) - \hat{\theta}(0)$$

remains bdd

$\Rightarrow \quad \int_0^\infty x^2(t)\,dt < \infty$

So, $x(t) \to 0$ as $t \to \infty$ by Barbalat's lemma.

Main Takeaways:

- the control law
$$\begin{cases} u = -k(\hat{\theta})x \\ \dot{\hat{\theta}} = x^2 \\ k(\hat{\theta}) = \hat{\theta} + 1) \end{cases}$$
is capable of stabilizing <u>any</u> plant of the form
$$\dot{x} = \theta x + u, \qquad \theta \in \mathbb{R}.$$
- Moreover, all signals in the closed-loop system remain bdd for all $t \geq 0$, w/ arbitrary initial condition.

The above controller is a <u>universal regulator</u> for the class
$$\{\dot{x} = \theta x + u \quad : \quad \theta \in \mathbb{R}\}.$$

- the notation $\hat{\theta}$ suggests estimating θ; what we are doing is rather exhaustive search through the space of controller gains until we find something that works.

Downside: possibly poor transient behavior.

Universal Regulators for Scalar Plants

Consider the class of plants
$$\{\dot{y} = ay + bu \quad : \quad a \in \mathbb{R},\ b \neq 0\}$$
(Before: $a \in \mathbb{R}$, $b = 1$)

Do universal regulators exist?

<u>Preview:</u> if $\text{sign}(b)$ is unknown, a universal regulator exists, but it is not simple.
(Thm)

Simple case: sign(b) known

[suppose $b > 0$, without loss of generality]

Claim: $u = -ky \qquad \dot{k} = y^2 \leftarrow$ tuning law

is a universal regulator

Closed-loop system:
$$\dot{y} = (a - bk)y$$
$$\dot{k} = y^2$$

$V(y) = \dfrac{y^2}{2}$ [cand. Lyapunov fcn]

$$\dot{V}(y) = y\dot{y} \qquad \text{— that is, } \frac{d}{dt}V(y(t))$$
$$= (a-bk)y^2 \qquad\qquad = (a - bk(t))y^2(t)$$
$$= (a-bk)\dot{k}$$

Integrate to get: $\boxed{\dfrac{y^2(t)}{2} = ak(t) - \dfrac{b}{2}k^2(t) + C}$

— same argument as earlier gives:

$$\sup_{t \geq 0} y^2(t) = 2 \sup_{t \geq 0} \left\{ ak(t) - \frac{b}{2}k^2(t) + C \right\} < \infty$$

$k(t)$ bdd $\Rightarrow$ $y(t)$ bdd

$\dot{y}(t) = (a - bk(t))y(t)$ bdd

$0 \leq \int_0^t y^2(s)\,ds$ bdd $\Rightarrow$ $\int_0^\infty y^2(t)\,dt < \infty$

[$y \in L^2$]

$\Rightarrow y(t) \to 0$ as $t \to \infty$ by Barbalat's lemma.