Problems to be handed in

1 Recall Barbalat's lemma: If $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous and the integral

$$\int_0^\infty f(t)\,\mathrm{d}t$$

exists and is finite, then $f(t) \xrightarrow{t\to\infty} 0$. Give a *detailed* proof of the corollary we have been using: If $x(t)$ and $\dot{x}(t)$ are both bounded and W is a continuous function taking nonnegative values, such that the integral

$$\int_0^\infty W(x(t))\,\mathrm{d}t$$

exists and is finite, then $W(x(t)) \xrightarrow{t\to\infty} 0$.

2 Consider a linear time-varying system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t)\\ y(t) &= C(t)x(t)\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the output, and $A(t), C(t)$ are matrix-valued functions of time. Let $\Phi(t, s)$, for $0 \le s \le t$, denote its *transition matrix* — it solves the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, s) = A(t)\Phi(t, s), \qquad t \ge s$$

with initial condition $\Phi(s, s) = I_n$ (the $n \times n$ identity matrix). In particular, $x(t) = \Phi(t, s)x(s)$ for any $0 \le s \le t$.

We say that this system is *Uniformly Completely Observable* (UCO) if there exist positive constants β_1, β_2, T, such that the observability Gramian

$$M(t_0, t_0 + T) := \int_{t_0}^{t_0+T} \Phi(t, t_0)^\top C(t)^\top C(t)\Phi(t, t_0)\,\mathrm{d}t$$

satisfies the matrix inequality

$$\beta_1 I_n \preceq M(t_0, t_0 + T) \preceq \beta_2 I_n$$

for all $t_0 \ge 0$. Prove that if the system is UCO, then $y(t) \xrightarrow{t\to\infty} 0$ implies $x(t) \xrightarrow{t\to\infty} 0$.

3 Consider the first-order scalar plant

$$\dot{y} = \theta f(y) + u,$$

where $\theta \in \mathbb{R}$ is unknown and f is a known function (the case $f(y) = y$ was considered in class). Design a controller that achieves output regulation, $y(t) \xrightarrow{t\to\infty} 0$, while keeping all signals in the closed-loop system bounded. Give a proof of universal regulation by choosing a suitable candidate Lyapunov function. You may assume that f is as well-behaved as needed to guarantee existence and uniqueness of closed-loop solutions for all $t \ge 0$.

4 Consider the first-order scalar plant

$$\dot{y} = ay + bu,$$

where $a \in \mathbb{R}$ and $b > 0$ are unknown parameters. In class, we have shown that the dynamic output feedback controller $u = -ky$ with tuning law $\dot{k} = y^2$ achieves universal regulation using the candidate Lyapunov function $V(y) = \frac{1}{2}y^2$. Give an alternative analysis based on the weak Lyapunov criterion and a suitable Lyapunov function $V(y, k)$ for the closed-loop system

$$\begin{aligned}\dot{y} &= (a - bk)y \\ \dot{k} &= y^2\end{aligned}$$