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Linear deterministic adaptive control: fundamental limitations?

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Abstract

This paper is concerned with the achievable performance of adaptive control algorithms. We show that when the only uncertainty is in the form of fixed parameter errors, then there exists an adaptive feedback law whose performance can be made arbitrarily close to that achievable when the system is a priori known. The result is not intended as a practical strategy. Instead, we use it to make the, perhaps obvious, point that meaningful results on performance of adaptive control algorithms must account for non-ideal factors including, at a minimum, noise, parameter time variations and unstructured uncertainty.

Keywords: Adaptive control; Identification

1. Introduction

Many results are available regarding fundamental limitations that apply to linear feedback systems. These results take several different forms, e.g.

- limitations on time domain transients (see for example [9]);
- frequency domain integrals on sensitivity (see for example [3,5]); and
- $\mathscr{L}_2$ cheap control results (see for example [11,12]).

A summary of some of these results is available in recent texts, see for example [4,5,14].

The above work on fundamental limitations has focused on the case when the model is a priori known. This raises the question, regarding the effect of modeling errors on these results. A very preliminary first step in this direction was discussed in [7]. In that paper, the interaction between a right half plane zero and an uncertain pole was analyzed.

The issue of performance limitations in the face of model uncertainty clearly overlaps with the area of adaptive control. Indeed, if we view adaptive control as providing a solution to the control of specific classes of uncertain systems, then the question of performance limitations can be rephrased as, "How good can an adaptive controller be?" or "Does there exist a computable lower bound on achievable performance of adaptive control algorithms, and if so, how closely do existing algorithms approach this bound?"

Such questions have been posed in the literature. See for example the comparison of Lyapunov-based adaptive controllers and certainty equivalence adaptive controllers for linear time-invariant systems, discussed in [8].

More recently, Anderson and Gevers [1] have posed several questions regarding fundamental performance of trade-off issues in adaptive control. The later paper takes an engineering viewpoint of adaptive control. The authors of the current paper share a similar

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prospective on adaptive control. Indeed, we suggest that, to be meaningful, comparisons of performance must rigorously account for non-ideal factors including (but not restricted to):

- time-varying parameters,
- unstructured model uncertainty,
- measurement noise,
- actuator limitations, and
- numerical issues.

To reinforce the need to consider these non-ideal factors, we will present below a very simple result which shows that, in the ideal case, one can design an adaptive control algorithm whose performance is arbitrarily close to that achievable when the model is a priori known.

2. Problem setup

We consider an ideal adaptive control setup of the type considered in the literature on deterministic model reference adaptive control (see e.g. [2,6,10,13]).

Thus, consider a linear time-invariant continuous time system having transfer function $G(s)=b(s)/a(s)$, having known denominator degree, n, but unknown coefficients in the polynomials $a(s)$ and $b(s)$. It is well known (see [6]) that an alternative representation of $G(s)$ is as follows:

$$y(t)=\frac{E(D)-a(D)}{E(D)}\,y(t)+\frac{b(D)}{E(D)}\,u(t)+\eta(t), \qquad (1)$$

where D is the derivative operator, $E(D)$ is an arbitrarily chosen polynomial with roots in the left half plane (representing the observer poles) and $\eta(t)$ satisfies $E(D)\eta(t)=0$ and represents the state observation error. Clearly, there exist $a,c>0$ such that

$$|\eta(t)|\leqslant c\mathrm{e}^{-at} \quad \forall t>0, \qquad (2)$$

where c depends on the initial conditions and *a is completely determined by the choice of $E(D)$*, and hence, can be made as large as we want. Another way of writing (1) is

$$y(t)=\varphi(t)^{\mathrm{T}}\theta_0+\eta(t), \qquad (3)$$

where

$$\varphi(t)^{\mathrm{T}}=\left[\frac{D^{n-1}}{E(D)}\,y(t),\frac{D^{n-2}}{E(D)}\,y(t),\ldots,\frac{1}{E(D)}\,y(t),\right.$$
$$\left.\frac{D^{n-1}}{E(D)}\,u(t),\ldots,\frac{1}{E(D)}\,u(t)\right], \qquad (4)$$

$$\theta_0^{\mathrm{T}}=[e_1-a_1,e_2-a_2,\ldots,$$
$$e_n-a_n,b_1,\ldots,b_n]. \qquad (5)$$

$\{a_i\}$ and $\{b_i\}$ are the unknown coefficients of the polynomials $a(s)$ and $b(s)$, respectively, and $\{e_i\}$ denotes the coefficients of the known polynomial $E(s)$.

Our strategy will be to place the system under fixed (but otherwise arbitrary) feedback control for an arbitrarily small time period, δ. We do not assume that the fixed controller is necessarily stabilizing, although we do note that a finite escape cannot occur since the signal growth is at most exponential. During the period δ, we apply an arbitrarily small external signal so that $\varphi(t)$ satisfies

$$\int_0^{\delta}\varphi(t)\varphi(t)^{\mathrm{T}}\,\mathrm{d}t>0. \qquad (6)$$

It is known that this can be achieved by proper choice of the external input $r(t)$ (e.g. a combination of sufficient number of sine waves).

With the model in (3) we can apply the well-known recursive least squares algorithm to estimate θ_0. In continuous time, the algorithm takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\theta}(t)=P(t)\varphi(t)e(t) \qquad (7)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,P(t)=-P(t)\varphi(t)\varphi(t)^{\mathrm{T}}P(t), \qquad (8)$$

where $\hat{\theta}(t)$ is the estimate of θ_0, $\hat{\theta}(0)$ and $P(0)$ are the algorithm initial conditions,

$$e(t)=y(t)-\varphi(t)^{\mathrm{T}}\hat{\theta}(t)$$
$$=-\varphi(t)^{\mathrm{T}}\tilde{\theta}(t)+\eta(t) \qquad (9)$$

and

$$\tilde{\theta}(t)=\hat{\theta}(t)-\theta_0. \qquad (10)$$

Our main result (presented in the next section) is that by choice of $E(D)$ and $P(0)$ the estimation error can be made arbitrarily small. Under these conditions, we see that, in an arbitrarily small time we can estimate the parameters to any desired degree of accuracy. Under these conditions one can design the feedback controller to achieve essentially the same performance as if the parameters had been known a priori. Thus, in this ideal scenario, the fundamental limitations on performance of adaptive controllers can be arbitrarily close to the achievable performance when the model is known.

3. Main result

Lemma 1. *For any $\delta > 0$ for which Eq. (6) is satisfied and $\varepsilon > 0$, there exist $K(\delta,\varepsilon) > 0$ and $E(D)$ such that if we choose in the RLS algorithm $P(0) = K(\delta,\varepsilon)I$ we obtain*

$$\|\tilde{\theta}(t)\| \leqslant \varepsilon(\|\tilde{\theta}(0)\| + c) \quad \forall t > \delta \tag{11}$$

with c as in (2).

Proof (This result is not surprising and may exist in the literature. For completeness, we present a formal proof of the result). We denote

$$M(\delta) = \int_0^\delta \varphi(t)\varphi(t)^{\mathrm{T}}\,\mathrm{d}t \tag{12}$$

and let $\underline{\lambda}(\delta)$ and $\bar{\lambda}(\delta)$ be the minimal and maximal eigenvalues of $M(\delta)$, respectively. Then, from Eq. (6), $\underline{\lambda}(\delta) > 0$ and $\bar{\lambda}(\delta) > 0$. Furthermore, from Eqs. (8) and (12) we obtain

$$P(t)^{-1} = P(0)^{-1} + M(t) \tag{13}$$

and from Eqs. (7) and (8)

$$P(t)^{-1}\tilde{\theta}(t) = P(0)^{-1}\tilde{\theta}(0) + \int_0^t \varphi(\sigma)\eta(\sigma)\,\mathrm{d}\sigma. \tag{14}$$

Then, choosing $P(0) = KI$, $K > 0$ we obtain

$$\begin{aligned}\tilde{\theta}(t) &= \frac{1}{K}P(t)\tilde{\theta}(0) + P(t)\int_0^t \varphi(\sigma)\eta(\sigma)\,\mathrm{d}\sigma \\ &= [I + KM(t)]^{-1}\tilde{\theta}(0) + v(t).\end{aligned} \tag{15}$$

We can readily see that $v(t)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -P(t)\varphi(t)\varphi(t)^{\mathrm{T}}v(t) + P(t)\varphi(t)\eta(t).$$

Then, defining the function

$$V(t) = v(t)^{\mathrm{T}}P(t)^{-1}v(t) + \int_t^\infty \eta(\sigma)^2\,\mathrm{d}\sigma,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = -(\varphi(t)^{\mathrm{T}}v(t) - \eta(t))^2 < 0,$$

which implies that

$$v(t)^{\mathrm{T}}P(t)^{-1}v(t) \leqslant V(t) \leqslant V(0) \leqslant \frac{c^2}{2a}.$$

On the other hand,

$$v(t)^{\mathrm{T}}P(t)^{-1}v(t) \geqslant v(t)^{\mathrm{T}}P(0)^{-1}v(t) = \frac{1}{K}\|v(t)\|^2.$$

Hence, we have

$$\|v(t)\|^2 \leqslant \frac{c^2K}{2a}. \tag{16}$$

Also, since $M(t) \geqslant M(\delta)$ for all $t \geqslant \delta$ we have

$$[I + KM(t)]^{-1} \leqslant [I + KM(\delta)]^{-1} \quad \forall t \geqslant \delta.$$

Combining this with (16) in (15) we conclude that

$$\begin{aligned}\|\tilde{\theta}(t)\| &\leqslant \|[I + KM(\delta)]^{-1}\|\,\|\tilde{\theta}(0)\| + \|v(t)\| \\ &\leqslant \frac{1}{1 + K\underline{\lambda}(\delta)}\|\tilde{\theta}(0)\| + c\sqrt{\frac{K}{2a}}.\end{aligned}$$

Choosing

$$K(\delta,\varepsilon) \geqslant \frac{1-\varepsilon}{\varepsilon\underline{\lambda}(\delta)}$$

and

$$a(\delta,\varepsilon) \geqslant \frac{K(\delta,\varepsilon)}{2\varepsilon^2}$$

will result in (11) and the proof is complete. □

4. Conclusions

The paper confirms that parameters of an initially unknown system can be estimated to an arbitrary degree of accuracy in an arbitrarily small time by the application of an arbitrarily small external signal. Thus, in the ideal case, the performance of an adaptive algorithm can always be made arbitrarily close to the performance achieved when the model is fully known. Of course, this result would not be used in practice. For example, the presence of noise and numerical errors will limit the accuracy of estimation in closed loop. Our point here is not to deny this but to reinforce the point that adaptive control algorithms must, inter alia, address non-ideal issues in a rigorous way if meaningful benchmarks for performance limits are to be analyzed.

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