Lecture VII: Convolution representation of continuous-time systems

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BME 171: Signals and Systems
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This lecture

Plan for the lecture:

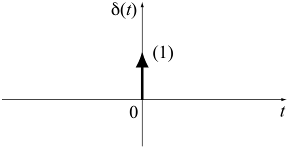
1. The unit impulse response
2. Derivation of the convolution representation of continuous-time LTI systems
3. Convolution of continuous-time signals
4. Causal LTI systems with causal inputs
5. Computing convolution integrals: examples

The unit impulse response

Let us consider a continuous-time LTI system

$$y(t) = S\Big\{x(t)\Big\}$$

and use the **unit impulse** $\delta(t)$ as input.



Let us define the **unit impulse response** of S as the corresponding output:

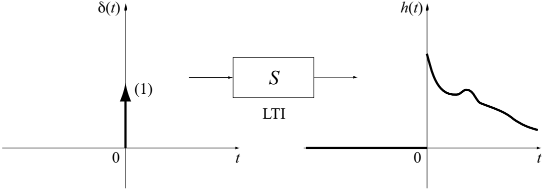
$$h(t) \triangleq S\Big\{\delta(t)\Big\}$$

We will now show that the output of S due to an **arbitrary** input $x(t)$ can be expressed in terms of $x(t)$ and $h(t)$.

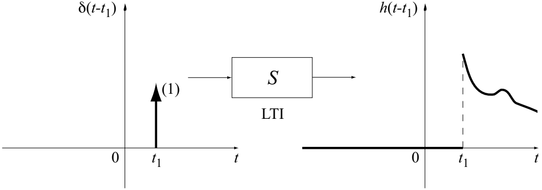
The shifted unit impulse response

Because S is time-invariant, the output due to a time-shifted unit impulse input is the time-shifted unit impulse response:

$$h(t) = S\Big\{\delta(t)\Big\}$$



$$h(t - t_1) = S\Big\{\delta(t - t_1)\Big\}$$



Derivation of the convolution representation

Using the sifting property of the unit impulse, we can write

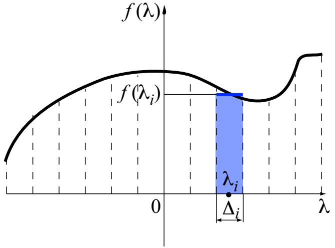
$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda$$

We will approximate the above integral by a sum, and then use linearity and time invariance of S to derive the convolution representation.

Given a function f, we have the following approximation:

$$\int_{-\infty}^{\infty} f(\lambda)d\lambda \approx \sum_i \Delta_i f(\lambda_i),$$

where the interval lengths Δ_i are sufficiently small.



Derivation of the convolution representation

For a fixed t, let us define

$$f(\lambda) \triangleq x(\lambda)\delta(t-\lambda).$$

Then we approximate the integral by the sum to get

$$\begin{aligned}
S\Big\{x(t)\Big\} &= S\Big\{\int_{-\infty}^{\infty} \underbrace{x(\lambda)\delta(t-\lambda)}_{=f(\lambda)}\, d\lambda\Big\} \\
&\approx S\Big\{\sum_i \Delta_i \underbrace{x(\lambda_i)\delta(t-\lambda_i)}_{=f(\lambda_i)}\Big\} \\
&= \sum_i S\left\{\Delta_i x(\lambda_i)\delta(t-\lambda_i)\right\} && \text{(by additivity)} \\
&= \sum_i \Delta_i x(\lambda_i) S\left\{\delta(t-\lambda_i)\right\} && \text{(by homogeneity)} \\
&= \sum_i \Delta_i x(\lambda_i) h(t-\lambda_i) && \text{(by def. of } h \text{ and time invariance)}
\end{aligned}$$

Derivation of the convolution representation

So far, we have shown that

$$S\Big\{x(t)\Big\} \approx \sum_i \Delta_i x(\lambda_i) h(t-\lambda_i)$$

Now let us define the function

$$g(\lambda) \triangleq x(\lambda)h(t-\lambda).$$

Then

$$S\Big\{x(t)\Big\} \approx \sum_i \Delta_i g(\lambda_i)$$

As we take $\max_i \Delta_i \to 0$, we obtain

$$\sum_i \Delta_i g(\lambda_i) \to \int_{-\infty}^{\infty} g(\lambda) d\lambda = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda.$$

Hence, we have the **convolution representation**

$$S\Big\{x(t)\Big\} = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

Convolution of continuous-time signals

Given two continuous-time signals $x(t)$ and $\nu(t)$, we define their **convolution** $x(t) \star \nu(t)$ as

$$x(t) \star \nu(t) \triangleq \int_{-\infty}^{\infty} x(\lambda)\nu(t-\lambda)d\lambda.$$

Just as in the discrete-time case, the convolution is

- **commutative:** $x(t) \star \nu(t) = \nu(t) \star x(t)$
- **associative:** $x(t) \star (\nu(t) \star \mu(t)) = (x(t) \star \nu(t)) \star \mu(t)$
- **distributive:** $x(t) \star (\nu(t) + \mu(t)) = x(t) \star \nu(t) + x(t) \star \mu(t)$
- **homogeneous:** $x(t) \star (a\nu(t)) = a(x(t) \star \nu(t))$ for any constant a

Thus, if $x(t)$ is an input to an LTI system S with unit impulse response $h(t)$, then the output $y(t)$ is given by

$$y(t) = S\{x(t)\} = x(t) \star h(t)$$

Additional properties of the convolution

1. **Derivative of a convolution:**

$$\frac{d}{dt}(x(t) \star \nu(t)) = \left(\frac{d}{dt}x(t)\right) \star \nu(t) = x(t) \star \left(\frac{d}{dt}\nu(t)\right)$$

$$\begin{aligned} \frac{d}{dt}(x(t) \star \nu(t)) &= \frac{d}{dt}\left(\int_{-\infty}^{\infty} x(\lambda)\nu(t-\lambda)d\lambda\right) \\ &= \int_{-\infty}^{\infty} x(\lambda)\frac{d}{dt}\nu(t-\lambda)d\lambda \\ &= x(t) \star \left(\frac{d}{dt}\nu(t)\right) \end{aligned}$$

2. **Convolution with a unit impulse**

$$x(t) \star \delta(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda = x(t)$$

which follows by the sifting property of the unit impulse.

Causal LTI systems with causal inputs

Just as in the discrete-time case, a continuous-time LTI system is causal if and only if its impulse response $h(t)$ is zero for all $t < 0$. If S is causal, then

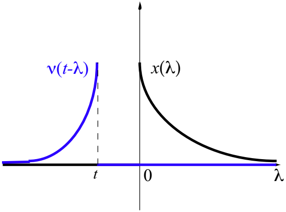
$$y(t) = S\Big\{x(t)\Big\} = \int_{-\infty}^{t} x(\lambda)h(t-\lambda)d\lambda$$

A signal $x(t)$ is **causal** if $x(t) = 0$ for $t < 0$. Then, if $h(t)$ is the unit impulse response of a causal LTI system, we have

$$y(t) = S\Big\{x(t)\Big\} = \int_{0}^{t} x(\lambda)h(t-\lambda)d\lambda$$

Computing convolution integrals: example 1

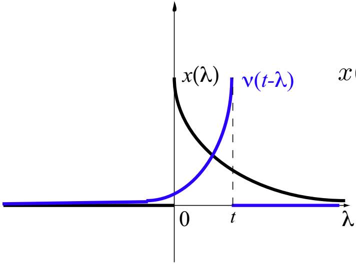
Let $x(t) = e^{-t}u(t)$, $\nu(t) = e^{-2t}u(t)$. Then $\nu(t-\lambda) = e^{2(\lambda-t)}u(t-\lambda)$.



Case 1: $t < 0$

$x(\lambda)\nu(t-\lambda) = 0$, so

$$x(t) \star \nu(t) = 0, \qquad t < 0$$



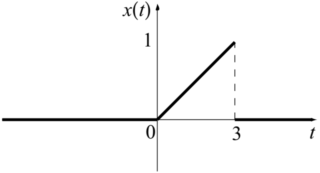
Case 1: $t \ge 0$

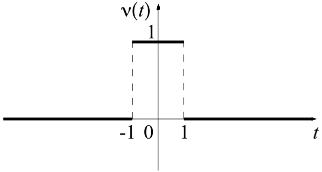
$$\begin{aligned} x(t) \star \nu(t) &= \int_{-\infty}^{\infty} e^{-\lambda} u(\lambda) u^{2(\lambda-t)} u(t-\lambda) d\lambda \\ &= \int_0^t e^{-\lambda} e^{2(\lambda-t)} d\lambda \\ &= e^{-2t} \left[e^{\lambda} \right]_0^t \\ &= e^{-t} - e^{-2t}, \qquad t \ge 0 \end{aligned}$$

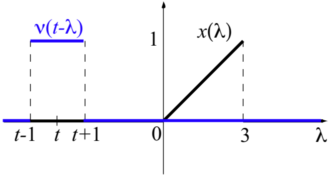
Hence, $x(t) \star \nu(t) = (e^{-t} - e^{-2t})u(t)$.

Computing convolution integrals: example 2

$$x(t) = \begin{cases} 0, & t < 0 \\ t/3, & 0 \le t \le 3 \\ 0, & t > 3 \end{cases} \qquad \nu(t) = \begin{cases} 0, & t < -1 \\ 1, & -1 \le t \le 1 \\ 0, & t > 1 \end{cases}$$





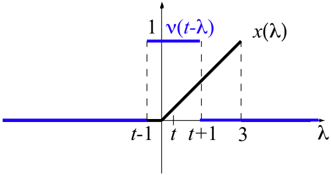


Case 1: $t < -1$

$x(\lambda)\nu(t-\lambda) = 0$, so

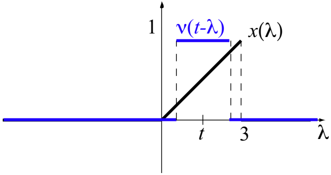
$$x(t) \star \nu(t) = 0, \qquad t < -1$$

Case 2: $-1 \le t \le 1$



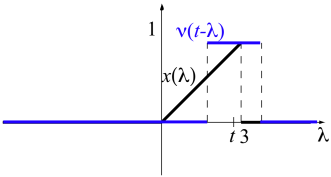
$$\begin{aligned} x(t) \star \nu(t) &= \int_0^{t+1} 1 \cdot \frac{\lambda}{3} d\lambda \\ &= \left[\frac{\lambda^2}{6}\right]_0^{t+1} \\ &= \frac{(t+1)^2}{6}, \qquad -1 \le t \le 1 \end{aligned}$$

Case 3: $1 \le t \le 2$



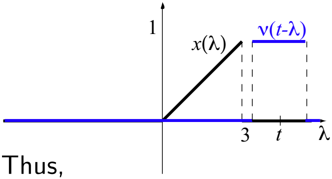
$$\begin{aligned} x(t) \star \nu(t) &= \int_{t-1}^{t+1} 1 \cdot \frac{\lambda}{3} d\lambda \\ &= \left[\frac{\lambda^2}{6}\right]_{t-1}^{t+1} \\ &= \frac{2t}{3}, \qquad 1 \le t \le 2 \end{aligned}$$

Computing convolution integrals: example 2



Case 4: $2 \le t \le 4$

$$\begin{aligned} x(t) \star \nu(t) &= \int_{t-1}^{3} 1 \cdot \frac{\lambda}{3} d\lambda \\ &= \left[\frac{\lambda^2}{6} \right]_{t-1}^{3} \\ &= -\frac{t^2 - 2t - 8}{6}, \qquad 2 \le t \le 4 \end{aligned}$$



Case 5: $t \ge 4$

$x(\lambda)\nu(t-\lambda) = 0$, so

$$x(t) \star \nu(t) = 0, \qquad t \ge 4.$$

Thus,

$$x(t) \star \nu(t) = \begin{cases} 0, & t < -1 \\ (t+1)^2/6, & -1 \le t \le 1 \\ 2t/3, & 1 \le t \le 2 \\ -(t^2 - 2t - 8)/6, & 2 \le t \le 4 \\ 0, & t > 4 \end{cases}$$