BME 171-02, Signals and Systems

Exam III: Solutions
100 points total

0. (5 pts.) Laplace transform tables.

1. (20 pts.) A signal $x(t)$ has Laplace transform

$$X(s) = \frac{s+2}{s^2+4s+5}$$

Find the Laplace transforms of the following signals without computing the inverse Laplace transform of $X(s)$:

(a) $y_1(t) = x(2t-1)u(2t-1)$

Solution:

$$Y_1(s) = \frac{1}{2}e^{-s/2}X\left(\frac{s}{2}\right) = \frac{1}{2}e^{-s/2}\frac{(s/2)+2}{(s/2)^2+4(s/2)+5} = \frac{s+4}{s^2+8s+20}e^{-s/2}$$

(b) $y_2(t) = e^{-3t}x(t)$

Solution:

$$Y_2(s) = X(s+3) = \frac{(s+3)+2}{(s+3)^2+4(s+3)+5} = \frac{s+5}{s^2+10s+26}$$

(c) $y_3(t) = x(t)\cos(7t)$

Solution:

$$Y_3(s) = \frac{1}{2}\left[X(s+7j)+X(s-7j)\right] = \frac{1}{2}\left[\frac{(s+7j)+2}{(s+7j)^2+4(s+7j)+5} + \frac{(s-7j)+2}{(s-7j)^2+4(s-7j)+5}\right]$$

(d) $y_4(t) = x(t) \star (x(t-1)u(t-1))$

Solution:

$$Y_4(s) = e^{-s}X^2(s) = e^{-s}\left(\frac{s+2}{s^2+4s+5}\right)^2 = e^{-s}\frac{s^2+4s+4}{s^4+8s^3+26s^2+40s+25}$$

2. (20 pts.) Obtain the inverse Laplace transforms of the following functions:

(a) $X(s) = \dfrac{5}{s^3 + s^2 + 9s + 9}$ (*Hint:* $s + 1$ is a factor.)

Solution:

$$X(s) = \frac{5}{s^2(s+1) + 9(s+1)} = \frac{5}{(s^2+9)(s+1)} = \frac{5}{(s+3j)(s-3j)(s+1)}$$

The poles are: $p_1 = -3j, p_2 = 3j, p_3 = -1$. Partial fraction expansion:

$$X(s) = \frac{c_1}{s+3j} + \frac{c_2}{s-3j} + \frac{c_3}{s+1}$$

Find the residues:

$$\begin{aligned}
c_1 &= [(s+3j)X(s)]_{s=-3j} = \left[\frac{5}{(s-3j)(s+1)}\right]_{s=-3j} = -\frac{5}{18+6j} = -\frac{1}{4} + \frac{1}{12}j \\
&= \sqrt{\frac{5}{72}} e^{j(\pi + \tan^{-1}(-1/3))} \\
c_2 &= \bar{c}_1 = -\frac{1}{4} - \frac{1}{12}j = \sqrt{\frac{5}{72}} e^{j(\pi + \tan^{-1}(-1/3))} \\
c_3 &= [(s+1)X(s)]_{s=-1} = \left[\frac{5}{(s+3j)(s-3j)}\right]_{s=-1} = \frac{5}{10} = \frac{1}{2}
\end{aligned}$$

$$x(t) = 2\sqrt{\frac{5}{72}} \cos\left(3t + \pi + \tan^{-1}\left(-\frac{1}{3}\right)\right) u(t) + \frac{1}{2} e^{-t} u(t)$$

(Note: $\tan^{-1}(-1/3) \approx -0.32$, so the phase is ≈ 2.82)

(b) $X(s) = \dfrac{s^2 + 5s + 7}{s^2 + 3s + 2}$

Solution:

$$X(s) = \frac{(s^2+3s+2) + (2s+5)}{s^2+3s+2} = 1 + \frac{2s+5}{s^2+3s+2} \equiv 1 + Y(s).$$

Decompose the second term into partial fractions:

$$Y(s) = \frac{2s+5}{s^2+3s+2} = \frac{2s+5}{(s+2)(s+1)} = \frac{c_1}{s+2} + \frac{c_2}{s+1}$$

Find the residues:

$$\begin{aligned}
c_1 &= [(s+2)Y(s)]_{s=-2} = \left[\frac{2s+5}{s+1}\right]_{s=-2} = -1 \\
c_2 &= [(s+1)Y(s)]_{s=-1} = \left[\frac{2s+5}{s+2}\right]_{s=-1} = 3
\end{aligned}$$

$$x(t) = \delta(t) - e^{-2t} u(t) + 3e^{-t} u(t)$$

3. (15 pts.) For each of the signals below, determine the poles of its Laplace transform without actually computing the Laplace transform.

   (a) $x_1(t) = \left[3e^{-2t} - te^{-2t} + 8e^{-t/3}\right] u(t)$

   Solution: $p_1 = p_2 = -2, p_3 = -1/3$

   (b) $x_2(t) = 4e^{-t}\cos(7t + \pi/4)u(t) + 9u(t)$

   Solution: $p_1 = -1 + 7j, p_2 = -1 - 7j, p_3 = 0$

   (c) $x_3(t) = 2\cos(3t - \pi/16)u(t) + 3\sin(4t)u(t)$

   Solution: $p_1 = 3j, p_2 = -3j, p_3 = 4j, p_3 = -4j$

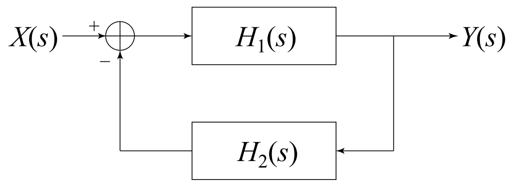
   (d) $x_4(t) = \left[e^{-3t}\cos(4t) + 2te^{-3t}\cos(4t + \pi/18)\right] u(t)$

   Solution: $p_1 = p_2 = -3 + 4j; p_3 = p_4 = -3 - 4j$

   (e) $x_5(t) = \left[e^{2t} + 3te^{2t} + e^{-t}\sin(6t)\right] u(t)$

   Solution: $p_1 = p_2 = 2, p_3 = -1 + 6j, p_4 = -1 - 6j$

4. (20 pts.) Consider the following system:



where

$$H_1(s) = \frac{s}{(s+1)(s+a)} \qquad \text{and} \qquad H_2(s) = \frac{b}{s}.$$

(a) Determine a and b such that the overall transfer function is

$$H(s) = \frac{s}{(s+4)(s+5)}$$

Solution:

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} = \frac{\frac{s}{(s+1)(s+a)}}{1 + \frac{s}{(s+1)(s+a)}\frac{b}{s}} = \frac{s}{(s+1)(s+a)+b} = \frac{s}{s^2+(a+1)s+(a+b)}$$

Since $(s+4)(s+5) = s^2 + 9s + 20$, we must have $a+1 = 9$ and $a+b = 20$, or

$$a = 8, \qquad b = 12$$

(b) Determine the output $y(t)$ of the system with the above transfer function to the unit-step input $x(t) = u(t)$.

Solution: $X(s) = 1/s$, so

$$Y(s) = H(s)X(s) = \frac{s}{(s+4)(s+5)}\frac{1}{s} = \frac{1}{(s+4)(s+5)}$$

Decompose into partial fractions:

$$Y(s) = \frac{c_1}{s+4} + \frac{c_2}{s+5}$$

Find the residues:

$$c_1 = [(s+4)Y(s)]_{s=-4} = \left[\frac{1}{s+5}\right]_{s=-4} = 1$$

$$c_2 = [(s+5)Y(s)]_{s=-5} = \left[\frac{1}{s+4}\right]_{s=-5} = -1$$

$$x(t) = (e^{-4t} - e^{-5t})u(t)$$

5. (20 pts.) Consider the signal

$$x(t) = \sum_{k=0}^{\infty} 2^{-k/2} \cos(40\pi k t)$$

(a) We want to design a lowpass filter that would remove no more than 10% of the signal energy. What should be the cutoff frequency of this filter?

Solution: $X(\omega) = \sum_{k=0}^{\infty} 2^{-k/2}\pi \left[\delta(\omega + 40\pi k) + \delta(\omega - 40\pi k)\right]$ By Parseval's theorem,

$$\begin{aligned}
\text{Signal energy} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\sum_{k=0}^{\infty} 2^{-k}\pi^2 \int_{-\infty}^{\infty} \left[\delta(\omega + 40\pi k) + \delta(\omega - 40\pi k)\right]^2 d\omega \\
&= \frac{\pi}{2}\sum_{k=0}^{\infty} 2^{-k} \cdot 2 \\
&= 2\pi.
\end{aligned}$$

If we use a lowpass filter to remove all frequencies higher than $40\pi k_0$, the filtered signal will be $\tilde{x}(t) = \sum_{k=0}^{k_0} 2^{-k/2}\cos(40\pi k t)$, and $\tilde{X}(\omega) = \sum_{k=0}^{k_0} 2^{-k/2}\pi \left[\delta(\omega + 40\pi k) + \delta(\omega - 40\pi k)\right]$.

$$\begin{aligned}
\text{Filtered signal energy} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{X}(\omega)|^2 d\omega \\
&= \frac{1}{2\pi}\sum_{k=0}^{k_0} 2^{-k}\pi^2 \int_{-\infty}^{\infty} \left[\delta(\omega + 40\pi k) + \delta(\omega - 40\pi k)\right]^2 d\omega \\
&= \pi \sum_{k=0}^{k_0} 2^{-k} \\
&= \pi \frac{1 - 2^{-(k_0+1)}}{1 - 1/2} \\
&= 2\pi\left(1 - 2^{-(k_0+1)}\right).
\end{aligned}$$

We want

$$\frac{\text{Filtered signal energy}}{\text{Signal energy}} = 1 - 2^{-(k_0+1)} \geq 0.9,$$

which is achieved when $k_0 \geq 4$. Thus, the cutoff frequency should be $40\pi k_0 = 160\pi$ rad/s.

(b) Let $\tilde{x}(t)$ be the corresponding filtered version of $x(t)$. What is the bandwidth of $\tilde{x}(t)$? What is the Nyquist rate?

Solution: the highest frequency component in $\tilde{x}(t)$ is $\omega = 160\pi$ rad/s. Thus, the bandwidth is 160π rad/s. The Nyquist rate is $2 \times$ bandwidth $= 320\pi$ rad/s.