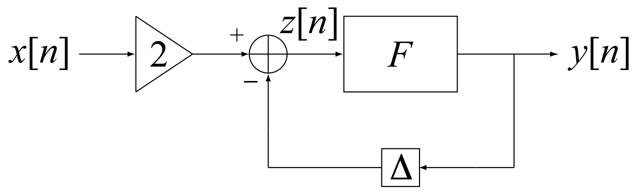
BME 171-02, Signals and Systems

Exam I: Solutions
100 points total

1. (15 pts.) Fill in the following table (for each column, give a "yes/no" answer and <u>briefly</u> justify it):

System	Linear	Time-invariant	Causal
$y(t) = 3x(t)\cos(t)$	yes	no	yes
$y(t) = \sqrt{x^2(t)}$	no	yes	yes
$y[n] = \begin{cases} +1, & x[n] \geq 0 \\ -1, & x[n] < 0 \end{cases}$	no	yes	yes
$y(t) = \int_t^{t+1} x(\lambda) d\lambda$	yes	yes	no
$y[n] = 2(x[n+1]u[n] - x[n]) + 1$	no	no	no

2. (15 pts.) Consider the following system:



Here, the system F is defined by the input-output relationship

$$F\Big\{z[n]\Big\} = z[n] - z[n-1],$$

and Δ is the unit delay

$$\Delta\Big\{w[n]\Big\} = w[n-1].$$

Write down the linear difference equation describing this system.

Solution. Let $z[n]$ be the output of the summer, as shown above. Then

$$y[n] = F\Big\{z[n]\Big\} = z[n] - z[n-1].$$

Now,

$$z[n] = 2x[n] - \Delta\Big\{y[n]\Big\} = 2x[n] - y[n-1].$$

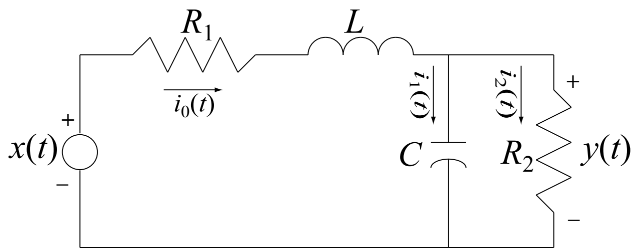
Therefore, substituting the expression for $z[n]$ into the first equation, we can write

$$\begin{aligned} y[n] &= z[n] - z[n-1] \\ &= \Big(\underbrace{2x[n] - y[n-1]}_{=z[n]}\Big) - \Big(\underbrace{2x[n-1] - y[n-2]}_{=z[n-1]}\Big) \\ &= 2x[n] - y[n-1] - 2x[n-1] + y[n-2]. \end{aligned}$$

Simplify to get

$$\boxed{y[n] + y[n-1] - y[n-2] = 2x[n] - 2x[n-1]}$$

3. (20 pts.) Consider the following circuit:



Write down the input-output differential equation for this circuit in terms of the input voltage $x(t)$ and the output voltage $y(t)$.

Solution. Let

$$\begin{aligned} i_0(t) &= \text{current through } R_1 \\ i_1(t) &= \text{current through } C \\ i_2(t) &= \text{current through } R_2 \end{aligned}$$

Then, using Kirchhoff's voltage law, we write

$$-x(t) + R_1 i_0(t) + L\frac{di_0(t)}{dt} + y(t) = 0.$$

On the other hand, using Kirchhoff's current law, we have

$$i_0(t) = i_1(t) + i_2(t).$$

Moreover,

$$i_1(t) = C\frac{dy(t)}{dt} \qquad \text{and} \qquad i_2(t) = \frac{y(t)}{R_2}.$$

Therefore,

$$i_0(t) = C\frac{dy(t)}{dt} + \frac{y(t)}{R_2}$$

and

$$\frac{di_0(t)}{dt} = C\frac{d^2y(t)}{dt^2} + \frac{1}{R_2}\frac{dy(t)}{dt}.$$

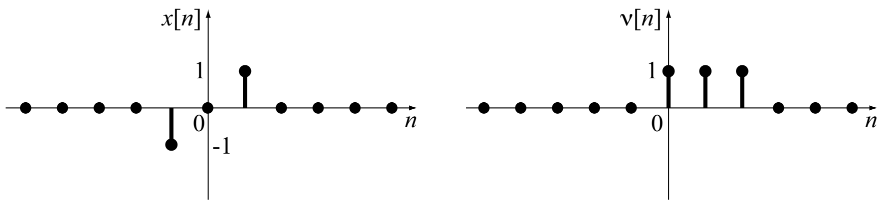
Substituting this into the KVL equation, we get

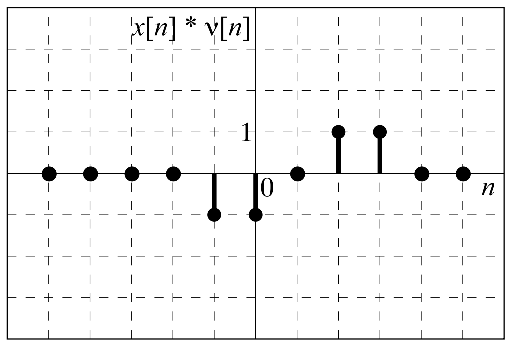
$$-x(t) + R_1\left(C\frac{dy(t)}{dt} + \frac{y(t)}{R_2}\right) + L\left(C\frac{d^2y(t)}{dt^2} + \frac{1}{R_2}\frac{dy(t)}{dt}\right) + y(t) = 0.$$

Simplify to get the final answer:

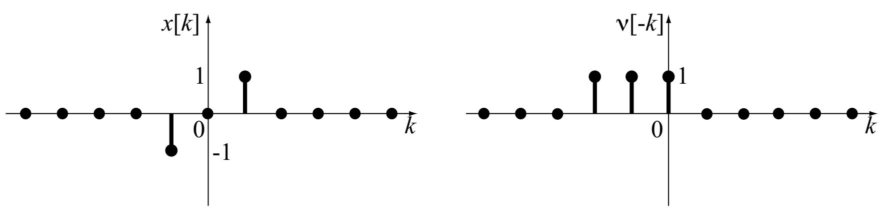
$$\boxed{LC\frac{d^2y(t)}{dt} + \left(R_1C + \frac{L}{R_2}\right)\frac{dy(t)}{dt} + \left(\frac{R_1}{R_2} + 1\right)y(t) = x(t).}$$

4. (25 pts.) Compute the convolution of the following two signals and plot the result on the set of axes provided. Show all your work!

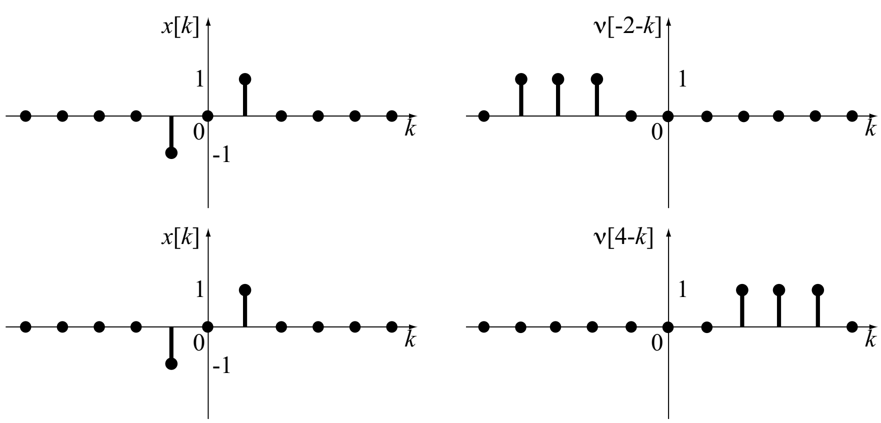




Solution. First, rename the time variable n into k. Next, flip and shift one of the signals. We will flip $\nu[k]$ to get $\nu[-k]$:



Now, shift by n to get $\nu[n-k]$. Note that there is no overlap between $x[k]$ and $\nu[n-k]$ as long as $n \leq -2$ or $n \geq 4$:



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So, $y[n] = x[n] \star \nu[n] = 0$ for $n \leq -2$ and for $n \geq 4$

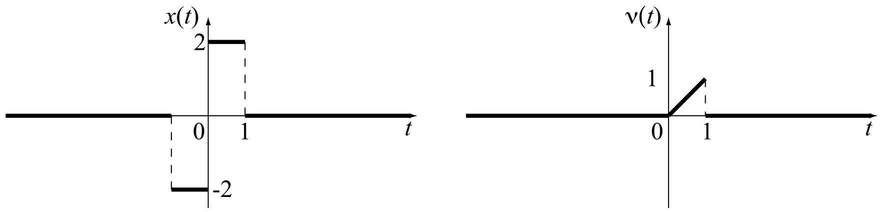
Now, shift and compute the overlap:

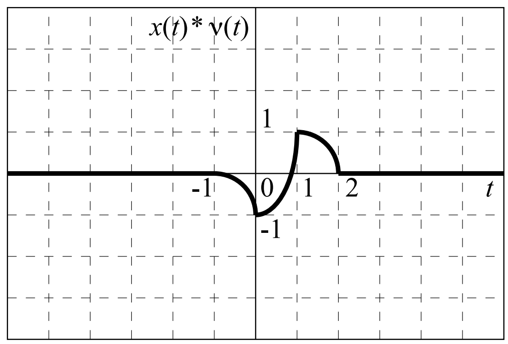
$$\begin{aligned}
y[-1] &= x[-1]\nu[0] + x[0]\nu[-1] + x[1]\nu[-2] \\
&= (-1)\cdot 1 + 0\cdot 0 + 1\cdot 0 \\
&= -1 \\
y[0] &= x[-1]\nu[1] + x[0]\nu[0] + x[1]\nu[-1] \\
&= (-1)\cdot 1 + 0\cdot 1 + 1\cdot 0 \\
&= -1 \\
y[1] &= x[-1]\nu[2] + x[0]\nu[1] + x[1]\nu[0] \\
&= (-1)\cdot 1 + 0\cdot 1 + 1\cdot 1 \\
&= 0 \\
y[2] &= x[-1]\nu[3] + x[0]\nu[2] + x[1]\nu[1] \\
&= (-1)\cdot 0 + 0\cdot 1 + 1\cdot 1 \\
&= 1 \\
y[3] &= x[-1]\nu[4] + x[0]\nu[3] + x[1]\nu[2] \\
&= (-1)\cdot 0 + 0\cdot 0 + 1\cdot 1 \\
&= 1
\end{aligned}$$

Overall,

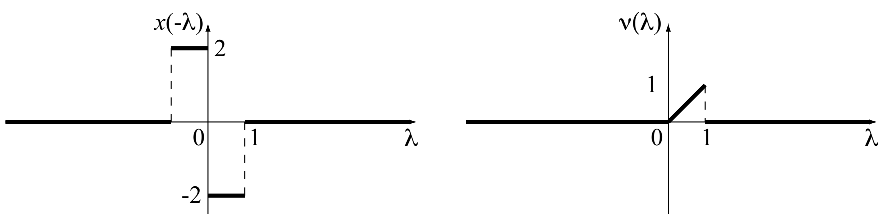
$$y[n] = x[n] \star \nu[n] = \begin{cases} 0, & n \leq -2 \\ -1, & n = -1, 0 \\ 0, & n = 1 \\ 1, & n = 2, 3 \\ 0, & n \geq 4 \end{cases}$$

5. (25 pts.) Compute the convolution of the following two signals. Write down its analytical form and sketch its plot in the set of axes provided. Show all your work!

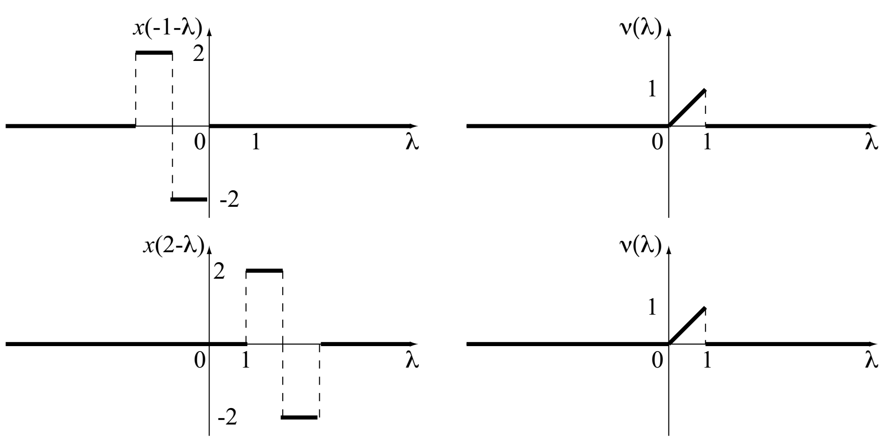




Solution. First, rename the time variable t into λ, say. Next, flip and shift one of the signals. We will flip $x(\lambda)$ to get $x(-\lambda)$:



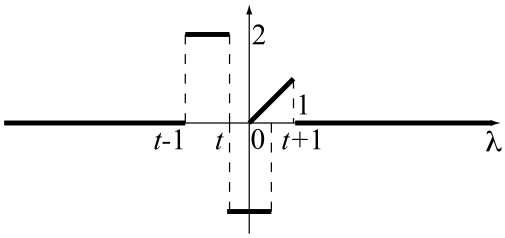
Now, shift by t to get $x(t-\lambda)$. Note that there is no overlap between $x(t-\lambda)$ and $\nu(\lambda)$ as long as $t \leq -1$ or $t \geq 2$:



So, $y(t) = x(t) \star \nu(t) = 0$ when $t \leq -1$ or when $t \geq 2$.

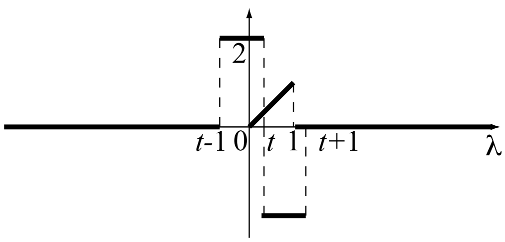
Now, we consider the remaining cases:

$-1 \leq t \leq 0$:



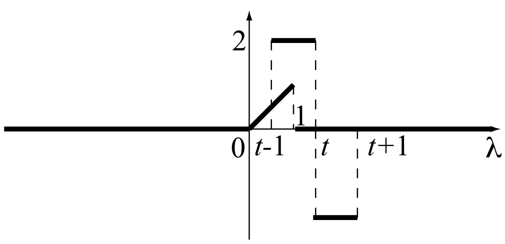
$$y(t) = \int_0^{t+1} (-2) \cdot \lambda d\lambda = -\left[\lambda^2\right]_0^{t+1} = -(t+1)^2$$

$0 \leq t \leq 1$:



$$y(t) = \int_0^{t} 2 \cdot \lambda d\lambda + \int_t^{1} (-2) \cdot \lambda d\lambda = \left[\lambda^2\right]_0^{t} - \left[\lambda^2\right]_t^{1} = t^2 - 1 + t^2 = 2t^2 - 1$$

$1 \leq t \leq 2$:



$$y(t) = \int_{t-1}^{1} 2 \cdot \lambda d\lambda = \left[\lambda^2\right]_{t-1}^{1} = 1 - (t-1)^2 = -t^2 + 2t$$

Overall,

$$\boxed{y(t) = x(t) \star \nu(t) = \begin{cases} 0, & t \leq -1 \\ -(t+1)^2, & -1 \leq t \leq 0 \\ 2t^2 - 1, & 0 \leq t \leq 1 \\ -t^2 + 2t, & 1 \leq t \leq 2 \\ 0, & t \geq 2 \end{cases}}$$