Required reading: Wright and Recht, Ch. 8.

1 (Exercise 1 in Wright and Recht, Ch. 8) Prove that, if f is convex and $x \in \mathrm{dom}(f)$, the subdifferential $\partial f(x)$ is closed and convex.

Reminder: A set $S \subseteq \mathbb{R}^n$ is closed if the limit of any convergent sequence $(v_n)_{n\geq 1}$ of elements of S is also an element of S:

$$v_n \in S \text{ for all } n = 1, 2, \dots \text{ and } v = \lim_{n\to\infty} v_n \text{ exists } \implies v \in S.$$

2 (Exercise 5 in Wright and Recht, Ch. 8) For the following norm functions f over $\mathbb{R}^n$, find the subdifferential $\partial f(x)$ and the directional derivative $f'(x, v)$ for all $x, v \in \mathbb{R}^n$:

(a) The ℓ_1 norm $f(x) = \|x\|_1 = \sum_{i=1}^n |x_i|$.

(b) The ℓ_∞ norm $f(x) = \|x\|_\infty = \max\limits_{1\leq i\leq n} |x_i|$.

(c) The ℓ_2 (Euclidean) norm $f(x) = \|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$.

3 (Exercise 7 in Wright and Recht, Ch. 8) Find the subdifferential $\partial f(x)$ of the piecewise-linear convex function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \max_{1\leq i\leq m} \left(a_i^T x + b_i\right),$$

where $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i = 1, \dots, m$.

4 Suppose that f is defined as a maximum of m convex, continuously differentiable functions; that is, $f(x) = \max_{1\leq i\leq m} f_i(x)$. Show that

$$\partial f(x) = \left\{ \sum_{i:\, f_i(x)=f(x)} \lambda_i \nabla f_i(x) : \lambda_i \geq 0, \sum_{i:\, f_i(x)=f(x)} \lambda_i = 1 \right\}.$$