Required reading: Wright and Recht, Ch. 2.

1 In this problem, you will prove the Cauchy–Schwarz inequality for vectors in $\mathbb{R}^n$: for $u = (u_1, \dots, u_n)^T$ and $v = (v_1, \dots, v_n)^T$,

$$|u^T v| \le \|u\| \|v\|,$$

where $\| \cdot \|$ stands for the ℓ_2 norm $\| \cdot \|_2$.

(a) Start by proving the following inequality:

$$|u^T v| \le \frac{1}{2\gamma}\|u\|^2 + \frac{\gamma}{2}\|v\|^2, \qquad \text{for all } \gamma > 0.$$

Hint: First prove it for $n = 1$, then go from there.

(b) Deduce the Cauchy–Schwarz inequality from the inequality proved in part (a).

2 (Exercise 2 from Wright and Recht, Ch. 2) Prove that the epigraph epi f is a convex subset of $\mathbb{R}^n \times \mathbb{R}$ for any convex function f.

3 (Exercise 5 from Wright and Recht, Ch. 2) Prove that no function $f : \mathbb{R}^n \to \mathbb{R}$ can be simultaneously strongly convex and Lipschitz.

4 (Exercise 7 from Wright and Recht, Ch. 2) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function with L-Lipschitz gradient and a minimizer x^* with function value $f^* = f(x^*)$.

(a) Show that, for any $x \in \mathbb{R}^n$, we have

$$f(x) - f^* \ge \frac{1}{2L}\|\nabla f(x)\|^2.$$

(b) Prove the following *co-coercivity* property: For all $x, y \in \mathbb{R}^n$,

$$[\nabla f(x) - \nabla f(y)]^T (x - y) \ge \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2.$$

Hint: Apply part (a) to the following two functions: $h_x(z) := f(z) - \nabla f(x)^T z$ and $h_y(z) := f(z) - \nabla f(y)^T z$.

5 (Exercise 8 in Wright and Recht, Ch. 2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an m-strongly convex function with L-Lipschitz gradient.

(a) Show that the function $q(x) := f(x) - \frac{m}{2}\|x\|^2$ is convex with $(L - m)$-Lipschitz gradient.

(b) By applying the co-coercivity property fron the previous problem to this function q, show that the following holds for all $x, y \in \mathbb{R}^n$:

$$[\nabla f(x) - \nabla f(y)]^T (x - y) \ge \frac{mL}{m + L}\|x - y\|^2 + \frac{1}{m + L}\|\nabla f(x) - \nabla f(y)\|^2.$$