This problem set, which will ***not*** be graded and which you do ***not*** have to turn in, is meant to give you a rough idea of the material from multivariable calculus and linear algebra you should be comfortable with in order to do well in ECE 490.

1 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function of n real variables $x_1, \dots, x_n$, differentiable in each of these variables. Prove that the following identity holds for all $x = (x_1, \dots, x_n)^T$ and $y = (y_1, \dots, y_n)^T$:

$$f(y) - f(x) = \int_0^1 \nabla f((1-t)x + ty)^T (y - x)\, \mathrm{d}t,$$

where $\nabla f(x) := \big(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\big)^T$ is the gradient of f evaluated at x.
Hint: With x and y fixed, consider the function $g(t) := f((1-t)x + ty)$ of a single real variable t. Use the fundamental theorem of calculus to write

$$g(1) - g(0) = \int_0^1 g'(t)\, \mathrm{d}t,$$

where $g'(t)$ denotes the derivative of g with respect to t.

2 Consider the quadratic function $f(x) = \frac{1}{2}x^T A x$, where x takes values in $\mathbb{R}^n$ and A is a given $n \times n$ matrix. Compute the gradient $\nabla f(x)$ and the Hessian $\nabla^2 f(x)$ (the $n \times n$ matrix of second-order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$).

3 Compute the gradient and the Hessian of the log-sum-exp function

$$f(x) = \log\left(\sum_{i=1}^n e^{x_i}\right),$$

where log denotes natural logarithm.

4 Let a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $b \in \mathbb{R}^m$ be given, such that $A^T A$ is invertible. Find the unique vector $x_* \in \mathbb{R}^n$ that minimizes the function

$$f(x) = \|Ax - b\|^2 = (Ax - b)^T (Ax - b)$$

and prove that it is, indeed, unique (i.e., $f(x) > f(x_*)$ for all $x \neq x_*$).

5 Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function, such that the inequality

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

holds for all vectors $x, y \in \mathbb{R}^n$. Prove that

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

for all $x, y \in \mathbb{R}^n$ and all $0 \leq t \leq 1$.

6 Let $e_1, \ldots, e_n$ denote the standard unit vectors in $\mathbb{R}^n$, i.e., the ith coordinate of e_i is equal to 1 and all other coordinates are equal to 0. Determine the eigenvalues of the matrix

$$A = e_1 e_1^T + 7 e_2 e_2^T - e_n e_n^T.$$

7 Let u and v be vectors in $\mathbb{R}^n$, such that $u^T v \neq 0$. Show that u is an eigenvector of the matrix $A = uv^T$ and find the corresponding eigenvalue.

8 The *singular values* of a matrix $A \in \mathbb{R}^{m \times n}$ are the nonnegative square roots of the eigenvalues of the symmetric matrix $A^T A$. Consider a *square* matrix $A \in \mathbb{R}^{n \times n}$ that can be written in the form

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T,$$

where σ_i are nonnegative real numbers and where $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are two orthonormal bases of $\mathbb{R}^n$, i.e.,

$$u_i^T u_j = \delta_{ij} \qquad \text{and} \qquad v_i^T v_j = \delta_{ij}.$$

Here, δ_{ij} is the Kronecker symbol, i.e., $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Find the singular values of A in terms of the σ_i's.