Online Learning Framework

- **Stat. learning:** (Vapnik, Haussler, &c)

  $z_1, z_2, \dots \overset{iid}{\sim} P$

  PAC: $z_1, \dots, z_n \longrightarrow \hat{f}_n$

  $$L(\hat{f}_n) - \inf_{f \in \mathcal{F}} L(f) \longrightarrow 0 \text{ as } n \to \infty$$

  uniformly in P

- **Online learning** w/o stochastic assumptions (Kivinen, Warmuth; Zinkevich '03, ...)

two-player interaction (game; see Cesa-Bianchi & Lugosi "Prediction, Learning, and Games")

Learner vs. Environment

$t = 1, 2, \dots, T$

at round t:

L selects $f_t \in \mathcal{F}$ ($\mathcal{F}$: closed, convex set in a Hilbert space $\mathcal{H}$)

E selects $z_t \in \mathcal{Z}$ (arbitrary space)

L incurs a loss $\ell(f_t, z_t)$, where

$$\ell : \mathcal{F} \times \mathcal{Z} \to \mathbb{R}$$

$$J_T(f^T) := \sum_{t=1}^{T} \ell_t(f_t) \qquad \ell_t(\cdot) := \ell(\cdot, z_t)$$

$$\begin{tikzcd} f_1 \arrow[r] \arrow[rr, bend right] \arrow[rrr, bend left] & z_1 \arrow[r] \arrow[rr, bend left] & f_2 \arrow[r] & z_2 \arrow[r] & \cdots \end{tikzcd}$$

L: f_t based on (f^{t-1}, z^{t-1})

E: z_t based on (f^t, z^{t-1})

$$\ell_t(f_t) = \ell(f_t, z_t)$$

best move: $f_t^* = \underset{f \in \mathcal{F}}{\operatorname{argmin}}\ \ell(f, z_t)$

(not realizable!)

Criterion: regret minimization

$$\underset{\substack{\uparrow \\ \text{regret}}}{R_T} := \underbrace{\sum_{t=1}^{T} \ell(f_t, z_t)}_{\substack{L\text{'s cost up} \\ \text{to } t=T}} - \underbrace{\inf_{f \in \mathcal{F}} \sum_{t=1}^{T} \ell(f, z_t)}_{\substack{\text{best possible cost} \\ \text{in hindsight}}}$$

$R_T = o(T)$

$$\boxed{\frac{1}{T} \min_{\substack{L\text{'s} \\ \text{strategies}}} \max_{\substack{E\text{'s} \\ \text{strategies}}} R_T \longrightarrow 0 \quad \text{as } T \to \infty}$$

- Online Convex Optimization (Zinkevich; Hazan, Abernethy, Rakhlin)

  - $\ell_t(\cdot)$ are convex + differentiable

- $\mathcal{F}$ is bdd, $D := \max\{\|f - f'\| : f, f' \in \mathcal{F}\} < \infty$
(diameter of $\mathcal{F}$)

- (ℓ_t) are L-Lipschitz: $\|\nabla \ell_t(\cdot)\| \leq L \quad \forall t$
- (ℓ_t) are m-strongly convex:
$$\ell_t(f') - \ell_t(f) - \langle \nabla \ell_t(f), f' - f \rangle \geq \frac{m}{2}\|f - f'\|^2$$
($m=0$: just convex)

Algorithm: online projected GD

$$f_{t+1} = \Pi(f_t - \alpha_t \nabla \ell_t(f_t))$$

$\ell_t(f_t) = \ell(f_t, z_t)$

where $\Pi : \mathcal{H} \to \mathcal{F}$ is the proj. onto $\mathcal{F}$
$(\alpha_t)_{t \geq 1}$: nonincr. step sizes $(\alpha_t > 0)$

Preview (w/ appropriate choice of α_t's):

$$R_T \sim \begin{cases} O(\sqrt{T}) & \text{for convex, } L\text{-Lip.} \\ O(\log T) & \text{for } m\text{-s.c., } L\text{-Lip.} \end{cases}$$

Analysis (sketch)

- potential fcn / Lyapunov fcn
- fix some $f^* \in \mathcal{F}$

0) $V_t := \|f_t - f^*\|^2$

$$V_{t+1} - V_t \leq (?)$$

1) $f_{t+1} = \Pi(\underbrace{f_t - \alpha_t \nabla \ell_t(f_t)}_{:= f^b_{t+1}})$

$f_{t+1} = \Pi(f^b_{t+1})$

$\|f^b_{t+1} - f^*\|^2 = \|f_t - f^* - \alpha_t g_t\|^2$ $\quad g_t := \nabla \ell_t(f_t)$

$= \|f_t - f^*\|^2 - 2\alpha_t \langle g_t, f_t - f^* \rangle + \alpha_t^2 \|g_t\|^2$

$= V_t - 2\alpha_t \langle g_t, f_t - f^* \rangle + \alpha_t^2 \|g_t\|^2$

$\leq V_t - 2\alpha_t \langle g_t, f_t - f^* \rangle + \alpha_t^2 L^2$

2) $V_{t+1} = \|f_{t+1} - f^*\|^2$

$= \|\Pi(f^b_{t+1}) - \Pi(f^*)\|^2$ $\quad f^* \in \mathcal{F} \Rightarrow \Pi(f^*) = f^*$

$\leq \|f^b_{t+1} - f^*\|^2$ $\quad \|\Pi(h) - \Pi(h')\| \leq \|h - h'\|$

$\leq V_t - 2\alpha_t \langle g_t, f_t - f^* \rangle + \alpha_t^2 L^2$

3) $2\langle g_t, f_t - f^* \rangle \leq \dfrac{V_t - V_{t+1}}{\alpha_t} + \alpha_t L^2$

by m-strong convexity (just convexity if $m=0$):

$2[\ell_t(f^*) - \ell_t(f_t)] \geq 2\langle g_t, f^* - f_t \rangle + m \underbrace{\|f_t - f^*\|^2}_{V_t}$

$2[\ell_t(f_t) - \ell_t(f^*)] \leq 2\langle g_t, f_t - f^* \rangle - m V_t$

$$\therefore\ 2[l_t(f_t) - l_t(f^*)] \leq \frac{V_t - V_{t+1}}{\alpha_t} - mV_t + \alpha_t L^2$$

where $$\frac{V_t - V_{t+1}}{\alpha_t} = \frac{V_t}{\alpha_t} - \frac{V_{t+1}}{\alpha_{t+1}} + \left(\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t}\right) V_{t+1}$$

4) sum from $t=1$ to $t=T$:

$$2\left[\sum_{t=1}^{T} l_t(f_t) - \sum_{t=1}^{T} l_t(f^*)\right]$$

$$\leq \sum_{t=1}^{T} \left(\frac{V_t}{\alpha_t} - \frac{V_{t+1}}{\alpha_{t+1}}\right) - m\sum_{t=1}^{T} V_t$$

$$\sum_{t=1}^{T} \frac{c}{\sqrt{t}} = O(\sqrt{T})$$

$$+ \sum_{t=1}^{T} \left(\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t}\right) V_{t+1} + L^2 \sum_{t=1}^{T} \alpha_t$$

Two cases: $m=0$ (convex) — (A)

$m>0$ (strongly convex) — (B)

(A): $$2\left(\sum_{t=1}^{T} l_t(f_t) - \sum_{t=1}^{T} l_t(f^*)\right)$$

$$\leq \sum_{t=1}^{T} \left(\frac{V_t}{\alpha_t} - \frac{V_{t+1}}{\alpha_{t+1}}\right) + \sum_{t=1}^{T} \overbrace{\left(\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t}\right)}^{\geq 0} \overbrace{V_{t+1}}^{\leq D^2}$$

$$+ L^2 \sum_{t=1}^{T} \alpha_t$$

$$\leq \frac{V_1}{\alpha_1} - \frac{V_{T+1}}{\alpha_{T+1}} + D^2\left(\frac{1}{\alpha_{T+1}} - \frac{1}{\alpha_1}\right) + L^2 \sum_{t=1}^{T} \alpha_t$$

$$\leq \cancel{\frac{D^2}{\alpha_1}} + \frac{D^2}{\alpha_{T+1}} - \cancel{\frac{D^2}{\alpha_1}} + L^2 \sum_{t=1}^{T} \alpha_t$$

$\alpha_t = \frac{c}{\sqrt{t}}$:

for any $f^* \in \mathcal{F}$,

$$\left.\begin{array}{l} \sum_{t=1}^{T} l_t(f_t) - \sum_{t=1}^{T} l_t(f^*) \leq c' \sqrt{T} \\ \quad \text{in particular, holds} \\ \left[\text{for } f^* = \underset{f \in \mathcal{F}}{\operatorname{argmin}} \sum_{t=1}^{T} l_t(\cdot) \right] \end{array}\right\} R_T = O(\sqrt{T})$$

Precise bound: $R_T \leq DL\sqrt{2T}$ $\left(\alpha_t = \frac{D}{L\sqrt{2t}}\right)$
(cf. lecture notes)

(B) $m > 0$: $\alpha_t = \frac{1}{mt}$

$$2\left[\sum_{t=1}^{T} l_t(f_t) - \sum_{t=1}^{T} l_t(f^*)\right]$$

$$\leq \sum_{t=1}^{T} \left(\frac{V_t}{\alpha_t} - \frac{V_{t+1}}{\alpha_{t+1}}\right) - m \sum_{t=1}^{T} V_t$$

$$+ \sum_{t=1}^{T} \left(\frac{1}{\alpha_{t+1}} - \frac{1}{\alpha_t}\right) V_{t+1} + L^2 \sum_{t=1}^{T} \alpha_t$$

$$= \frac{V_1}{\alpha_1} - \frac{V_{T+1}}{\alpha_{T+1}} - mV_1 - \underline{m \sum_{t=1}^{T-1} V_{t+1}}$$

$$+\underbrace{\sum_{t=1}^{T}\left(\frac{1}{\alpha_{t+1}}-\frac{1}{\alpha_t}\right)V_{t+1}} + L^2\sum_{t=1}^{T}\alpha_t$$

$$= \frac{V_1}{\alpha_1} - \frac{V_{T+1}}{\alpha_{T+1}} + \sum_{t=1}^{T-1}\left(\frac{1}{\alpha_{t+1}}-\frac{1}{\alpha_t}-m\right)V_{t+1}$$

$$-mV_1 + \left(\frac{1}{\alpha_{T+1}}-\frac{1}{\alpha_T}\right)V_{T+1} + L^2\sum_{t=1}^{T}\alpha_t$$

$$\leq D^2\underbrace{\left(\frac{1}{\alpha_T}-mT\right)}_{=0} + L^2\sum_{t=1}^{T}\alpha_t$$

$$= L^2\sum_{t=1}^{T}\alpha_t = \frac{L^2}{m}\sum_{t=1}^{T}\frac{1}{t} \leq \frac{L^2}{m}(1+\log T).$$

Remark:

1) convex, Lip.: simpler argument when T fixed
— choose const. α, then optimize

2) need decaying rates for s.-c.