Problems to be handed in

For the first two problems, you will need the definitions of positive semidefinite and positive definite matrices: A square matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite (respectively, positive definite) if $x^T A x \geq 0$ for all nonzero $x \in \mathbb{R}^n$ (respectively, $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$). We use the notation $A \succeq 0$ and $A \succ 0$ to indicate that $A$ is positive semidefinite (respectively, positive definite).

1  (convex and strictly convex functions) A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if, for any two points $x, y \in \mathbb{R}^n$ and any $\lambda \in [0, 1],$

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$  

and strictly convex if the inequality in (1) is strict unless $x = y$ or $\lambda \in \{0, 1\}$.

(i) Prove that $f$ is convex (respectively, strictly convex) if and only if the function $\lambda \mapsto f(\lambda x + (1 - \lambda)y)$ of $\lambda \in [0, 1]$ is convex (respectively, strictly convex) for any fixed pair of points $x, y \in \mathbb{R}^n$.

(ii) Suppose that $f$ is twice differentiable. Use the result of part (i) to show that $f$ is convex (respectively, strictly convex) if its Hessian $\nabla^2 f(x)$ is positive semidefinite (respectively, positive definite) at every point $x \in \mathbb{R}^n$.

(iii) Let $f$ be a strictly convex function which is bounded from below, i.e., there exists some $c \in \mathbb{R}$ such that $f(x) \geq c$ for all $x$. Prove that any minimizer of $f$, if it exists, is unique. Give an example of a strictly convex function bounded from below that does not have any minimizers.

(iv) Consider the function

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) := x^T A x + x^T v + \alpha$$  

for some $A \in \mathbb{R}^{n \times n}$, $v \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Prove that $f$ is convex (respectively, strictly convex and bounded from below) if and only if $A \succeq 0$ (respectively, $A \succ 0$).

(v) Use the result of part (iv) to give an alternative proof of the ‘completion-of-squares’ lemma.

2  (Schur complements and the LQR) In this problem, we will give the full rigorous derivation of the optimal controller and the value functions for the linear quadratic regulator.

(i) Let $X$ be a block matrix of the form

$$X = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$  

where $A, B, C$ are matrices of appropriate shapes, and $A$ and $C$ are both symmetric (note that this implies that $X$ itself is symmetric, $X = X^T$). Prove that, if $C \succ 0$, then $X \succeq 0$ if and only if $S := A - B C^{-1} B^T \succeq 0$ (the matrix $S$ is called the Schur complement of $C$ in $X$).
(ii) Let the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{m \times m}$ be given, such that $C > 0$, and consider the function

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad f(x,u) := x^T Ax + u^T Cu + 2x^T Bu. \quad (4)$$

Fix $x \in \mathbb{R}^n$ and consider the problem of minimizing $f(x,u)$ over $u \in \mathbb{R}^m$. Prove that $u^* = -C^{-1}B^Tx$ is the unique minimizer and that

$$f^*(x) := \min_{u \in \mathbb{R}^m} f(x,u) = x^T Sx,$$  \quad (5)

where $S$ is the Schur complement of $C$ in the matrix $X$ defined in (3). Prove that if $X \succeq 0$, then $f^*(x) = x^TSx$ is a convex function of $x$.

(iii) Use the above results to give a complete solution of the dynamic program for the LQR problem

$$X_{t+1} = AX_t + BU_t + W_t,$$  \quad (6)

where the disturbances $W_0, W_1, \ldots$ are i.i.d. zero mean random vectors with covariance matrix $\Sigma = \mathbb{E}[W_tW_t^T]$ and the costs are given by

$$c_0(x,u) = \ldots = c_{T-1}(x,u) = x^T Qx + u^T Ru, \quad c_T(x) = x^T Q_T x \quad (7)$$

with $Q, Q_T \succeq 0$ and $R > 0$. In particular, show that the optimal controller is linear, i.e.,

$$g_t^*(x) = G_t x$$

for some matrices $G_t \in \mathbb{R}^{m \times n}$, and the value functions are convex quadratic, i.e.,

$$V_t(x) = x^T K_t x + \alpha_t$$

with $K_t \succeq 0$ and $\alpha_t \geq 0$.

**Hint:** Use the Schur complement condition to establish convexity of $V_t$. In particular, first show that $V_t \geq 0$ and then use this to show that $K_t$ must be positive semidefinite.

3 (hidden Markov models) Consider a discrete-time stochastic process $\{(X_t, Y_t)\}_{t=0}^\infty$, where $X_t$ and $Y_t$ take values in finite sets $\mathcal{X}$ and $\mathcal{Y}$, respectively. We say that this process is a hidden Markov model if, for each $t$ and for all tuples $x_0 \in \mathcal{X}_0$ and $y_0 \in \mathcal{Y}_0$,

$$\mathbb{P}[(X_t, Y_t) = (x_t, y_t)| (X_0, Y_0) = (x_0, y_0), \ldots, (X_{t-1}, Y_{t-1}) = (x_{t-1}, y_{t-1})] = \mathbb{P}[(X_t, Y_t) = (x_t, y_t)| (X_{t-1}, Y_{t-1}) = (x_{t-1}, y_{t-1})],$$  \quad (8)

and

$$\mathbb{P}[Y_0 = y_0, \ldots, Y_t = y_t| X_0 = x_0, \ldots, X_t = x_t] = \prod_{s=0}^t \mathbb{P}[Y_t = y_t| X_t = x_t].$$  \quad (9)

The process $\{X_t\}_{t \geq 0}$ is called the hidden state process or the signal process, while $\{Y_t\}_{t \geq 0}$ is the observation process. In other words, (8) states that $\{(X_t, Y_t)\}_{t \geq 0}$ is a Markov chain with state space $\mathcal{X} \times \mathcal{Y}$, while (9) states that the observations $Y_t$ are conditionally independent given the corresponding signals $X'_t$. 

2
(i) Prove that a hidden Markov model is completely specified by the probability law $\mu$ of $X_0$ and by two sequences of nonnegative matrices $\{P(t)\}_{t \geq 0}$ and $\{M(t)\}_{t \geq 0}$, where the rows and columns of $P(t)$ are indexed by the elements of $\mathcal{X}$ and the entries are given by

$$P(t)(x, x') = P[X_{t+1} = x' | X_t = x],$$

while the rows (respectively, columns) of $M(t)$ are indexed by $\mathcal{X}$ (respectively, by $\mathcal{Y}$) and the entries are given by

$$M(t)(x, y) = P[Y_t = y | X_t = x].$$

(ii) Let $\{(X_t, Y_t)\}_{t \geq 0}$ be a discrete-time stochastic process, where each $(X_t, Y_t)$ takes values in the Cartesian product $\mathcal{X} \times \mathcal{Y}$ of two finite sets, $X_0 \sim \mu$, and there exist sequences of functions $\{f_t\}_{t \geq 0}$ and $\{h_t\}_{t \geq 0}$ and two mutually independent sequences $\{W_t\}_{t \geq 0}$ and $\{V_t\}_{t \geq 0}$ of i.i.d. random variables$^1$ that are also independent of $X_0$, such that

$$X_{t+1} = f_t(X_t, W_t) \quad \text{and} \quad Y_t = h_t(X_t, V_t) \quad \text{for all } t \geq 0. \quad (12)$$

Prove that $\{(X_t, Y_t)\}_{t \geq 0}$ is a hidden Markov model.

(iii) Consider a hidden Markov model specified by $\mu$, $\{P(t)\}_{t \geq 0}$, and $\{M(t)\}_{t \geq 0}$. Prove that it admits a realization of the form described in part (ii), i.e., one can always find two mutually independent sequences $\{W_t\}_{t \geq 0}$ and $\{V_t\}_{t \geq 0}$ of i.i.d. random variables that are also independent of $X_0$, as well as sequences of functions $\{f_t\}_{t \geq 0}$ and $\{h_t\}_{t \geq 0}$, such that $X_0 \sim \mu$ and

$$X_{t+1} = f_t(X_t, W_t) \quad \text{and} \quad Y_t = h_t(X_t, V_t) \quad \text{for all } t \geq 0. \quad (13)$$

(iv) Let $\{(X_t, Y_t)\}_{t \geq 0}$ be a hidden Markov model. Prove that the signal process $\{X_t\}_{t \geq 0}$ is a Markov chain. (On the other hand, $\{Y_t\}_{t \geq 0}$ may not be a Markov chain.)

4 (unnormalized nonlinear filter) Let $\{(X_t, Y_t)\}_{t \geq 0}$ be a hidden Markov model specified by $\mu$, $\{P(t)\}_{t \geq 0}$, and $\{M(t)\}_{t \geq 0}$. Let $y_0, y_1, \ldots$ be a fixed sequence of observations. For each $t \geq 0$, define the unnormalized filtering distributions

$$\sigma_t(x_t) := \sum_{x_0^{-1} \in X_0^{-1}} \mu(x_0) \prod_{s=0}^{t-1} P(s)(x_s, x_{s+1}) \prod_{s=0}^{t} M(s)(x_s, y_s). \quad (14)$$

(i) Prove that the unnormalized filtering distributions can be computed recursively according to

$$\sigma_{t+1}(x_{t+1}) = \sigma_t P(t)(x_{t+1}) M(t+1)(x_{t+1}, y_{t+1}) \quad (15)$$

with the initial condition

$$\sigma_0(x_0) = \mu(x_0) M(0)(x_0, y_0). \quad (16)$$

$^1$The distributions of $W_0$ and $V_0$ need not be the same; in fact, $W_0$ and $V_0$ need not take values in the same set.
(ii) Prove that the (normalized) filtering distributions $\pi_t$ can be computed in terms of the unnormalized filtering distributions as

$$
\pi_t(x_t) = \frac{\sigma_t(x_t)}{\sum_{x \in \mathcal{X}} \sigma_t(x)}. 
$$

(17)