Note: Problems (or parts of problems) marked with a star (★) are required for graduate students to receive 4 credit hours; undergraduate students who solve these problems will receive extra credit points.

Submission: Write your name, netid, and U for undergrad/G for grad in the upper right-hand corner of the first page of your written solutions. Typewritten solutions will receive 5 extra credit points.

Problems to be handed in

1 Let $X = (X_t)_{t\in\mathbb{R}}$ be a weakly stationary stochastic signal with zero mean and autocorrelation function $R_X(\tau) = \sigma^2 e^{-|\tau|}$. Let Y be the stochastic signal obtained from X via

$$Y_t = \frac{1}{2t_0}\int_{t-t_0}^{t+t_0} X_t \mathrm{d}t,$$

where $t_0 > 0$ is a fixed window size.

(a) Prove that Y is also weakly stationary.

(b) Find the mean m_Y and the power spectral density S_Y.

2 In this problem, we look at the Doppler effect due to a signal source moving with random velocity. For simplicity, we assume that the motion takes place in one dimension. Consider an observer located at the origin and a signal source initially located at some fixed point at distance r from the origin. The source emits a complex exponential signal with amplitude a and angular frequency ω_0; the stochastic signal received by the observer can be represented by the complex exponential

$$X_t = a\exp\left[\mathrm{i}\omega_0\left(t - \frac{r+Vt}{c}\right)\right],$$

where $c > 0$ is the speed of signal propagation. Here, the velocity V is a random variable with pdf f_V. The autocorrelation function for a complex-valued stochastic signal X is defined as $R_X(s,t) \triangleq \mathbf{E}[X_s^* X_t]$, where the star denotes complex conjugation. Prove $R_X(t, t+\tau)$ depends only on τ and compute its "power spectral density" S_X. The quotes around "power spectral density" are necessary because the mean of X_t changes with t, so X is nonstationary. Nevertheless, since in this case $R_X(t,t+\tau) = R_X(\tau)$, the Fourier inversion formula gives

$$\mathbf{E}[|X_t|^2] = R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\mathrm{d}\omega$$

for any t, so it still makes sene to think about S_X as the power spectral density of X.

Hint: The formula for S_X should involve a, c, ω_0, and f_V.

3 In this problem, we will explore some properties of jointly Gaussian random variables.

(a) Recall that the characteristic function of a scalar random variable X is given by

$$\Phi_X(u) = \mathbf{E}[e^{\mathrm{i}uX}], \qquad u \in \mathbb{R}.$$

and that the joint characteristic function of a random vector $X = (X_t)_{t\in\{1,\dots,n\}}$ is given by

$$\Phi_{X_1,\dots,X_n}(u_1,\dots,u_n) = \mathbf{E}[e^{\mathrm{i}(u_1X_1+\dots+u_nX_n)}], \qquad u_1,\dots,u_n \in \mathbb{R}.$$

Assume that $X_1,\dots,X_n$ have a joint pdf $f_{X_1,\dots,X_n}$. We say that $X_1,\dots,X_n$ are independent random variables if

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\dots f_{X_n}(x_n),$$

where f_{X_i} denotes the marginal pdf of X_i. Prove that $X_1,\dots,X_n$ are independent if and only if

$$\Phi_{X_1,\dots,X_n}(u_1,\dots,u_n) = \Phi_{X_1}(u_1)\Phi_{X_2}(u_2)\dots\Phi_{X_n}(u_n)$$

for all $u_1,\dots,u_n \in \mathbb{R}$.

(b) We say that $X_1,\dots,X_n$ are uncorrelated if the covariance matrix C_X is diagonal, i.e., if $C_X(s,t) = \mathbf{E}[X_sX_t] - \mathbf{E}[X_s]\mathbf{E}[X_t] = 0$ for $s \neq t$. In general, uncorrelated random variables can still be dependent. Use the result from part (a) to prove that if $X_1,\dots,X_n$ are uncorrelated and jointly Gaussian, then they are independent.

(c) Let X be a Gaussian random vector. In class, we have proved that the projection a^TX of X onto any deterministic vector $a = (a_1,\dots,a_n)^T \in \mathbb{R}^n$ is Gaussian. Now consider an $m\times n$ matrix $A = (A_{ij})_{i,j\in\{1,\dots,n\}}$ and form the random vector $Y = AX$. Prove that Y is also a Gaussian random vector.

(d) (★) Let X be a Gaussian random variable with mean 0 and variance σ^2. Let U be a Rademacher random variable (i.e., $\mathbf{P}[U = \pm 1] = \frac{1}{2}$) independent of X. Prove that $Y = UX$ is also Gaussian with mean 0 and variance σ^2, but X and Y are *not* jointly Gaussian.

Hint: Consider the sum $X + Y$.

4 Let X be a zero-mean stationary Gaussian stochastic signal. Compute the crosscorrelation function $R_{XY}(\tau)$ between X and Y, where $Y_t = g(X_t)$ with the following choices for g:

(a) The full-wave rectifier $g(x) = |x|$.

(b) The power-law detector $g(x) = x^p$ for $p \in \mathbb{N}$.

(c) The gating function $g(x) = u(t+1) - u(t-1)$.