Note: Problems (or parts of problems) marked with a star (⋆) are required for graduate students to receive 4 credit hours; undergraduate students who solve these problems will receive extra credit points.

Submission: Write your name, netid, and U for undergrad/G for grad in the upper right-hand corner of the first page of your written solutions. Typewritten solutions will receive 5 extra credit points.

Problems to be handed in

1 Let $X = (X_t)_{t \in \mathbb{Z}_+}$ be a finite-state Markov chain with probability transition matrix M. Suppose that M has an invariant distribution π (i.e., $\pi M = \pi$). Prove that X is *strongly* stationary if and only if the initial state X_0 has distribution π.

Hint: Pass to a suitable imperative description $X_{t+1} = f(X_t, U_t)$.

2 Let $N = (N_t)_{t \ge 0}$ be a Poisson process with rate λ. Consider the following stochastic signal $X = (X_t)_{t \ge 0}$ with state space $\mathsf{X} = \{-1, +1\}$:

$$X_t = \begin{cases} +1, & \text{if } N_t \text{ is even} \\ -1, & \text{if } N_t \text{ is odd} \end{cases}.$$

(a) Sketch the typical path of X.

(b) Find the probability distribution of X_t, i.e., $\mathbf{P}[X_t = \pm 1]$.

Hint: You may want to look up power-series representations for sinh and cosh.

(c) Find the mean function $m_X(t) = \mathbf{E}[X_t]$.

(d) Find the autocorrelation function $R_X(s,t) = \mathbf{E}[X_s X_t]$.

Hint: The case $s = t$ is simple. For $t > s$, first compute the conditional probabilities $\mathbf{P}[X_t = \pm 1 | X_s = \pm 1]$.

(e) Is X weakly stationary? Why or why not?

3 Consider the random signal $X = (X_t)_{t \in \mathbb{R}}$ given by $X_t = A \cos \omega t + B \sin \omega t$, where A and B are two jointly distributed real-valued random variables. In class, we have proved that if X is WS, then:

1. $\mathbf{E}[A] = \mathbf{E}[B] = 0$ (both A and B have zero mean).
2. $\mathrm{Var}[A] = \mathrm{Var}[B] = \sigma^2$ (A and B have the same variance).
3. $\mathbf{E}[AB] = 0$ (A and B are uncorrelated).

Prove that if A and B satisfy these three conditions, then X is WS.

4 In this problem, we will explore some properties of the Wiener process. A *standard Wiener process* $W = (W_t)_{t\geq 0}$ is a Wiener process with $D = 1$.

(a) Prove that the covariance function of W is given by $C_X(s,t) = \min\{s,t\}$.

(b) Let $c > 0$ be a fixed positive constant, and define another stochastic signal $Y = (Y_t)_{t\geq 0}$ by letting $Y_t = \frac{1}{\sqrt{c}} W_{ct}$. Prove that Y is also a standard Wiener process. (This shows that the sample paths of a Wiener process look the same at every time scale — as long as we rescale space to compensate for the time scaling.)

(c) Again, let $c > 0$ be a fixed constsant, and define another stochastic signal $Z = (Z_t)_{t\geq 0}$ by letting $Z_t = W_{t+c} - W_c$. Prove that Z is a standard Wiener process, and that it is independent of $(W_t)_{0\leq t\leq c}$. (This shows that the Wiener process can be thought of continually restarting anew from its current position.)

(d) ($\star$) For $b > 0$, define the *hitting time*

$$\tau_b \triangleq \min\{t \geq 0 : W_t \geq b\},$$

i.e., the first time when the particle is at a distance b away from the origin (it may, and will, go below b later, and then above b, and then below, and so on). This is a random variable, since it depends on the random path of W_t. You will prove the following neat formula:

$$\mathbf{P}[\tau_b \leq t] = 2Q\left(\frac{b}{\sqrt{t}}\right), \qquad t \geq 0$$

where $Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-x^2/2} \mathrm{d}x$ is the complementary Gaussian cdf.

(i) By the law of total probability,

$$\mathbf{P}[\tau_b \leq t] = \mathbf{P}[\tau_b \leq t, W_t \leq b] + \mathbf{P}[\tau_b \leq t, W_t > b].$$

Now argue that the events $\{\tau_b \leq t, W_t > b\}$ and $\{W_t > b\}$ are equivalent (the continuity of W_t as a function of t is crucial for this to hold), and conclude from this that

$$\mathbf{P}[\tau_b \leq t] = \mathbf{P}[W_t \leq b | \tau_b \leq t] \mathbf{P}[\tau_b \leq t] + Q\left(\frac{b}{\sqrt{t}}\right).$$

(ii) Again, using the continuity of W_t in t, argue that $\mathbf{P}[W_t \leq b | \tau_b \leq t] = \frac{1}{2}$ (it may be helpful to draw a picture).

(iii) Put all the pieces together to obtain the formula we seek.

5 ($\star$) Let $N = (N_t)_{t \geq 0}$ be a Poisson process with rate λ, and $T = (T_k)_{k \in \mathbb{Z}_+}$ be the arrival times of N (with $T_0 = 0$). Let M be a given $n \times n$ Markov matrix. Consider a continuous-time stochastic signal $X = (X_t)_{t \geq 0}$ with finite state space $\mathsf{X} = \{0, \ldots, n-1\}$ that evolves as follows: it starts from $X_0 = 0$ and stays the same until the next arrival, at which point it changes randomly to a different state with probabilities prescribed by M. That is, $X_t = 0$ for $t < T_1$; then at $t = T_1$, $X_t = y$ with probability $M(X_0, y) = M(0, y)$, for each $y \in \mathsf{X}$. Then the state X_t stays the same until $t = T_2$, at which point it changes randomly to a new state y' with probability $M(X_{T_1}, y')$, etc.

(a) Prove that X is a Markov process.

(b) Let p_t denote the probability distribution of X_t, i.e., $p_t(x) = \mathbf{P}[X_t = x]$ for each $x \in \mathsf{X}$. Prove the following explicit formula for p_t:

$$p_t = p_0 e^{\lambda t (M - I_n)},$$

where I_n is the $n \times n$ identity matrix, p_0 is the initial state distribution (in this case, $p_0(x) = 1$ if $x = 0$ and 0 otherwise), and the matrix exponential e^A for a square matrix A is defined as

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Hint: Use the fact that the number of state transitions between times 0 and t is equal to N_t, the number of arrivals by time t, then apply the law of total probability.

(c) Consider the binary case $\mathsf{X} = \{0, 1\}$ with $M(0,0) = M(1,1) = \frac{1}{2}$. Compute the matrix $e^{\lambda t (M - I_2)}$ explicitly. What can you say about the long-term behavior of p_t – i.e., will it converge to a limiting distribution, and, if the answer is "yes," how fast is the convergence?