Problems to be handed in

1 Let $X = (X_t)_{t \in \mathbb{Z}}$ be a finite-state Markov chain with probability transition matrix $M$. Suppose that $M$ has an invariant distribution $\pi$ (i.e., $\pi M = \pi$). Prove that $X$ is strongly stationary if and only if the initial state $X_0$ has distribution $\pi$.

*Hint:* Pass to a suitable imperative description $X_{t+1} = f(X_t, U_t)$.

2 Let $N = (N_t)_{t \geq 0}$ be a Poisson process with rate $\lambda$. Consider the following stochastic signal $X = (X_t)_{t \geq 0}$ with state space $X = \{-1, +1\}$:

$$X_t = \begin{cases} +1, & \text{if } N_t \text{ is even} \\ -1, & \text{if } N_t \text{ is odd} \end{cases}$$

(a) Sketch the typical path of $X$.

(b) Find the probability distribution of $X_t$, i.e., $P[X_t = \pm 1]$.

*Hint:* You may want to look up power-series representations for sinh and cosh.

(c) Find the mean function $m_X(t) = \mathbb{E}[X_t]$.

(d) Find the autocorrelation function $R_X(s,t) = \mathbb{E}[X_t X_s]$.

*Hint:* The case $s = t$ is simple. For $t > s$, first compute the conditional probabilities $P[X_t = \pm 1|X_s = \pm 1]$.

(e) Is $X$ weakly stationary? Why or why not?

3 Consider the random signal $X = (X_t)_{t \in \mathbb{R}}$ given by $X_t = A \cos \omega t + B \sin \omega t$, where $A$ and $B$ are two jointly distributed real-valued random variables. In class, we have proved that if $X$ is WS, then:

1. $\mathbb{E}[A] = \mathbb{E}[B] = 0$ (both $A$ and $B$ have zero mean).

2. $\text{Var}[A] = \text{Var}[B] = \sigma^2$ ($A$ and $B$ have the same variance).

3. $\mathbb{E}[AB] = 0$ ($A$ and $B$ are uncorrelated).

Prove that if $A$ and $B$ satisfy these three conditions, then $X$ is WS.
4 In this problem, we will explore some properties of the Wiener process. A standard Wiener process $W = (W_t)_{t \geq 0}$ is a Wiener process with $D = 1$.

(a) Prove that the covariance function of $W$ is given by $C_X(s, t) = \min\{s, t\}$.

(b) Let $c > 0$ be a fixed positive constant, and define another stochastic signal $Y = (Y_t)_{t \geq 0}$ by letting $Y_t = \frac{1}{\sqrt{c}} W_{ct}$. Prove that $Y$ is also a standard Wiener process. (This shows that the sample paths of a Wiener process look the same at every time scale — as long as we rescale space to compensate for the time scaling.)

(c) Again, let $c > 0$ be a fixed constant, and define another stochastic signal $Z = (Z_t)_{t \geq 0}$ by letting $Z_t = W_{t+c} - W_c$. Prove that $Z$ is a standard Wiener process, and that it is independent of $(W_t)_{0 \leq t \leq c}$. (This shows that the Wiener process can be thought of continually restarting anew from its current position.)

(d) For $b > 0$, define the hitting time

$$\tau_b = \min\{t \geq 0 : W_t \geq b\},$$

i.e., the first time when the particle is at a distance $b$ away from the origin (it may, and will, go below $b$ later, and then above $b$, and then below, and so on). This is a random variable, since it depends on the random path of $W_t$. You will prove the following neat formula:

$$P[\tau_b \leq t] = 2Q\left(\frac{b}{\sqrt{t}}\right), \quad t \geq 0$$

where $Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-x^2/2} dx$ is the complementary Gaussian cdf.

(i) By the law of total probability,

$$P[\tau_b \leq t] = P[\tau_b \leq t, W_t \leq b] + P[\tau_b \leq t, W_t > b].$$

Now argue that the events $\{\tau_b \leq t, W_t > b\}$ and $\{W_t > b\}$ are equivalent (the continuity of $W_t$ as a function of $t$ is crucial for this to hold), and conclude from this that

$$P[\tau_b \leq t] = P[W_t \leq b]P[\tau_b \leq t] + Q\left(\frac{b}{\sqrt{t}}\right).$$

(ii) Again, using the continuity of $W_t$ in $t$, argue that $P[W_t \leq b | \tau_b \leq t] = \frac{1}{2}$ (it may be helpful to draw a picture).

(iii) Put all the pieces together to obtain the formula we seek.
(★) Let $N = (N_t)_{t \geq 0}$ be a Poisson process with rate $\lambda$, and $T = (T_k)_{k \in \mathbb{Z}_+}$ be the arrival times of $N$ (with $T_0 = 0$). Let $M$ be a given $n \times n$ Markov matrix. Consider a continuous-time stochastic signal $X = (X_t)_{t \geq 0}$ with finite state space $X = \{0, \ldots, n-1\}$ that evolves as follows: it starts from $X_0 = 0$ and stays the same until the next arrival, at which point it changes randomly to a different state with probabilities prescribed by $M$. That is, $X_t = 0$ for $t < T_1$; then at $t = T_1$, $X_t = y$ with probability $M(X_0, y) = M(0, y)$, for each $y \in X$. Then the state $X_t$ stays the same until $t = T_2$, at which point it changes randomly to a new state $y'$ with probability $M(X_{T_1}, y')$, etc.

(a) Prove that $X$ is a Markov process.

(b) Let $p_t$ denote the probability distribution of $X_t$, i.e., $p_t(x) = P[X_t = x]$ for each $x \in X$. Prove the following explicit formula for $p_t$:

$$p_t = p_0 e^{\lambda t (M - I_n)},$$

where $I_n$ is the $n \times n$ identity matrix, $p_0$ is the initial state distribution (in this case, $p_0(x) = 1$ if $x = 0$ and 0 otherwise), and the matrix exponential $e^A$ for a square matrix $A$ is defined as

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

*Hint:* Use the fact that the number of state transitions between times 0 and $t$ is equal to $N_t$, the number of arrivals by time $t$, then apply the law of total probability.

(c) Consider the binary case $X = \{0, 1\}$ with $M(0, 0) = M(1, 1) = \frac{1}{2}$. Compute the matrix $e^{\lambda t (M - I_2)}$ explicitly. What can you say about the long-term behavior of $p_t$ – i.e., will it converge to a limiting distribution, and, if the answer is “yes,” how fast is the convergence?