(1) Let $X_1, \ldots, X_n$ be $n$ independent Bernoulli($\theta$) random variables. Prove the following multiplicative Chernoff bound:

$$\Pr \left( \frac{1}{n} \sum_{i=1}^{n} X_i \geq (1 + \gamma) \mu \right) \leq e^{-\gamma^2 n \mu / 3}$$

for any $0 \leq \gamma \leq 1$ and any $\theta \leq \mu \leq 1$. You may need the fact that

$$\sum_{k \geq (1+\gamma)\mu n} \binom{n}{k} \mu^k (1 - \mu)^{n-k} \leq e^{-\gamma^2 n \mu / 3}.$$

(2) Let $X_1, \ldots, X_n$ be $n \geq 2$ real-valued random variables (not necessarily independent). Suppose that there exists some constant $\sigma > 0$, such that for all $s > 0$ and all $i = 1, \ldots, n$

$$E \left[ e^{s X_i} \right] \leq e^{s^2 \sigma^2 / 2}$$

Prove the following:

(a) $E \left[ \max_{1 \leq i \leq n} X_i \right] \leq \sigma \sqrt{2 \log n}$.

**Hint.** Try to bound $e^{s \Pr[\max_{1 \leq i \leq n} X_i]}$, exploit convexity of the function $\phi(x) = e^{sx}$.

(b) Assuming that $X_i \geq 0$ for all $i = 1, \ldots, n$,

$$\Pr \left( \max_{1 \leq i \leq n} X_i \geq \sqrt{2 \sigma^2 n} \right) \leq \sqrt{\frac{\log n}{n}}.$$

(3) Let $X^n = (X_1, \ldots, X_n)$ be an $n$-tuple of independent random variables taking values in some space $X$. The *Hamming distance* between any $n$-tuples $x^n, y^n \in X^n$ is defined as the number of coordinates in which $x^n$ and $y^n$ differ:

$$d(x^n, y^n) \triangleq \sum_{i=1}^{n} 1_{\{x_i \neq y_i\}}.$$

If $B$ is an arbitrary (measurable) subset of $X^n$, then the Hamming distance between an $n$-tuple $x^n \in X^n$ and $B$ is defined as

$$d(x^n, B) \triangleq \min_{y^n \in B} d(x^n, y^n).$$

Use McDiarmid’s inequality to prove the following fact: if

$$\epsilon \geq \sqrt{\frac{1}{2n} \log \frac{1}{\Pr(B)}},$$
then
\[ \mathbb{P}(d(X^n, B) \geq n\varepsilon) \leq \exp \left( -2n \left( \varepsilon - \sqrt{\frac{1}{2n} \log \frac{1}{\mathbb{P}(B)}} \right)^2 \right), \]

where \( \mathbb{P}(B) = \mathbb{P}(X^n \in B) \) is the probability of the set \( B \in X^n \) under the joint distribution of \( X^n \).

**Hint.** You may find the following fact handy: \( d(x^n, B) \leq 0 \) if and only if \( x^n \in B \).