ECE 515 / ME 540 **Assignment # 6**

Issued: Nov 9 Due: Nov 16, 2023

Reading Assignment:

BMP, Ch. 10.

Problems:

1. Consider the optimal control problem

$$\ddot{x} = u, \qquad J(u) = \int_{t_0}^{t_1} \Big(x^4(t) + u^2(t) \Big) \, \mathrm{d}t + x^3(t_1).$$

   (i) Write down a partial differential equation for the optimal cost V, and a boundary condition for it.

   (ii) Simplify the PDE by computing the minimum in it. Using this minimum calculation, write down an expression for the optimal control law in state feedback form. (This expression can contain partial derivatives of the optimal cost, evaluated along the optimal trajectory.)

2. Consider the LQR problem

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ J &= \int_{t_0}^{t_1} (x_2^2 + u^2) \, \mathrm{d}t \end{aligned}$$

   Write down the Riccati differential equation (with its boundary condition) and the expressions for the optimal cost V and the optimal state feedback control u^* (these expressions will depend on the solution P to the Riccati equation, but you don't need to compute this solution).

3. Consider the infinite-horizon LQR problem

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ J &= \int_{0}^{\infty} (x_2^2 + u^2) dt \end{aligned}$$

   Find the optimal cost V and the optimal control u^* in state feedback form. Show that the closed-loop system is stable but not asymptotically stable. Which condition of the theorem that guarantees closed-loop asymptotic stability is violated?

4. Consider the infinite-horizon LQR problem

$$\begin{aligned} \dot{x} &= ax + bu \\ J &= \int_{0}^{\infty} (qx^2 + ru^2) \, \mathrm{d}t \end{aligned}$$

where x and u are scalar, a, q, r are positive, and b is arbitrary. Suppose that a, b, q are fixed but r can vary.

(i) Show that in the limit as $r \to \infty$ (the "expensive control" case) the optimal control yields the closed-loop dynamics $\dot{x} = -ax$, in other words, the eigenvalue of the optimal closed-loop system tends to $-a$, the opposite of the open-loop eigenvalue.

(ii) Show that in the limit as $r \to 0$ (the "cheap control" case) the eigenvalue of the optimal closed-loop system moves off to $-\infty$.