Reading Assignment:
BMP, Ch. 4–6.

Problems:

1. Consider a system \( \dot{x} = f(x) \). Suppose that \( \frac{d}{dt}(x(t)TPx(t)) \leq -x(t)^TQx(t) \), where \( P \) and \( Q \) are symmetric positive definite matrices. Prove that, under this condition, the system is exponentially stable, in the sense that its solutions satisfy \( |x(t)| \leq ce^{-\mu t}|x(0)| \) for some \( c, \mu > 0 \). Note that this statement is true whether the system is linear or not.

Hint: use the inequality
\[
\lambda_{\text{min}}(M)|x|^2 \leq x^TMx \leq \lambda_{\text{max}}(M)|x|^2,
\]
where \( \lambda_{\text{min}}(M) \) and \( \lambda_{\text{max}}(M) \) are, respectively, the largest and the smallest eigenvalues of the matrix \( M \). You may also use the fact that if a function \( v(t) \) satisfies \( \dot{v} \leq -av \), then \( v(t) \leq ce^{-at}v(0) \).

2. Show that, if all eigenvalues of a matrix \( A \) have real parts strictly less than some \( -\mu < 0 \), then for every \( Q = Q^T > 0 \) the equation \( PA + A^TP + 2\mu P = -Q \) has a unique solution \( P = P^T > 0 \). Show that in this case the solutions of the LTI system \( \dot{x} = Ax \) satisfy \( |x(t)| \leq ce^{-\mu t}|x(0)| \) for some \( c > 0 \). (The number \( \mu \) is called a stability margin.)

3. (corrected) Recall that, for an LTI system \( \dot{x} = Ax + Bu \), the controllability Gramian has the form
\[
W(0,t) = \int_0^t e^{-A\tau}BB^T e^{-A^T\tau}d\tau.
\]

(i) Prove that the matrix
\[
\bar{W}(0,t) := \int_0^t e^{A\tau}BB^T e^{A^T\tau}d\tau.
\]
is nonsingular for some \( t > 0 \) if and only if \( W(0,t) \) is.

(ii) The result of (i) implies that the pair \((A, B)\) is controllable if and only if the pair \((-A, B)\) is controllable. Is this true for LTV systems? Prove or give a counterexample.

4. Consider the LTI system
\[
\begin{align*}
\dot{x} &= Ax \\
y &= Cx
\end{align*}
\]
and suppose that the eigenvalues of $A$ have negative real parts. Consider the function $V(x) = x^T M x$, where $M$ denotes the observability Gramian for the infinite time horizon, i.e.,

$$M = M(0, \infty) = \int_0^\infty e^{A^T t} C^T Ce^{A t} \, dt.$$

Show that, along the solutions of the system, we have

$$\dot{V} = -|y|^2.$$

5. Construct minimal (i.e., controllable and observable) realizations of the following transfer functions:

$$\frac{s - 3}{s^2 - 5s + 6}, \quad \frac{s^2 + 1}{s^3 - 2s^2 + s}.$$