Stability: Weak Lyapunov Functions

Recap:

$\dot{x} = \theta x + u$ $\quad (\theta \in \mathbb{R}$, unknown$)$

goal: output regulation $\quad x(t) \to 0$ as $t \to \infty$
$\quad u(t)$ bounded $\forall t \geq 0$

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} (\theta - \hat{\theta} - 1)x \\ x^2 \end{pmatrix} \qquad u = -(\hat{\theta}+1)x$$

Candidate Lyapunov fcn: $V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\hat{\theta} - \theta)^2$

$\dot{V} = -x^2 \leq 0$

Equilibria: $(x, \hat{\theta})$ s.t. $\begin{cases} (\theta - \hat{\theta} - 1)x = 0 \\ x^2 = 0 \end{cases}$

$\Rightarrow x = 0, \ \hat{\theta} \in \mathbb{R}$

all points on the line $\{(0, \hat{\theta}) : \hat{\theta} \in \mathbb{R}\}$ are equilibria

$\dot{V}(0, \hat{\theta}) = 0$ for any $\hat{\theta} \in \mathbb{R}$

— not enough info to conclude $x(t) \to 0$!

Consider an autonomous system

$\dot{x} = f(x) \qquad x(0) = x_0 \in \mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}^n$ continuous

$f(0) = 0$ — equilibrium at 0

Candidate Lyapunov fcn $V : \mathbb{R}^n \to [0, \infty)$, V is C^1

Thm (weak Lyapunov stability criterion)

Suppose $\exists$ a continuous fcn $W : \mathbb{R}^n \to [0, \infty)$ s.t.

$\dot{V}(x) \leq -W(x) \leq 0 \qquad$ (for all $x \in \mathbb{R}^n$)

Then, for every solution $x(t)$ <u>that remains bounded</u>

for all $t \geq 0$, $W(x(t)) \longrightarrow 0$ as $t \to \infty$.

Remarks:

1) boundedness of $x(t)$ is important!
2) if V is radially unbounded ($V(x) \to \infty$ as $|x| \to \infty$) then all solutions are bdd automatically. Khalil
3) V is not required to be p.d. ($V(x)$ can still be 0 even if $x \neq 0$)
4) Krasovskii-LaSalle thm: every bdd trajectory approaches the largest positively invariant set inside the set $\{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$

A set $D \subset \mathbb{R}^n$ is positively invariant if

$$x(0) \in D \implies x(t) \in D \quad \forall t \geq 0$$

If $D \subseteq \{x : \dot{V}(x) = 0\}$ is the largest p.inv. set, then

$$x(t) \text{ bdd} \implies \lim_{t\to\infty} \text{dist}(x(t), D) = 0.$$

Proof

(will need a lemma along the way)

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))\,ds \qquad \dot{V} \leq -W$$

$$\leq V(x(0)) - \int_0^t W(x(s))\,ds$$

$$\Rightarrow \int_0^t W(x(s))\,ds \leq V(x(0)) - V(x(t)) \qquad V \geq 0$$

$$\leq V(x(0)) < \infty$$

$$\forall t \geq 0: \; 0 \leq \int_0^t W(x(s))\,ds \leq V(x(0)) < \infty \qquad W \geq 0$$

$$\Rightarrow \quad \boxed{\int_0^\infty W(x(t))\,dt = \lim_{t\to\infty}\int_0^t W(x(s))\,ds \quad \text{exists and is finite}}$$

We would like to conclude that $W(x(t)) \to 0$.
Cannot do this in general!

Lemma (Barbalat's lemma) Let $f:[0,\infty)\to[0,\infty)$ be a uniformly continuous fcn, s.t.

$$\int_0^\infty f(t)\,dt = \lim_{t\to\infty}\int_0^t f(x(s))\,ds$$

exists and is finite. Then $f(t)\to 0$ as $t\to\infty$.

Remarks:

1) Continuity vs. uniform continuity

f is continuous if $\forall t, \forall \varepsilon>0$ $\exists\ \delta = \delta(\varepsilon, t) > 0$ s.t. $|t'-t|<\delta \Rightarrow |f(t')-f(t)|<\varepsilon$.

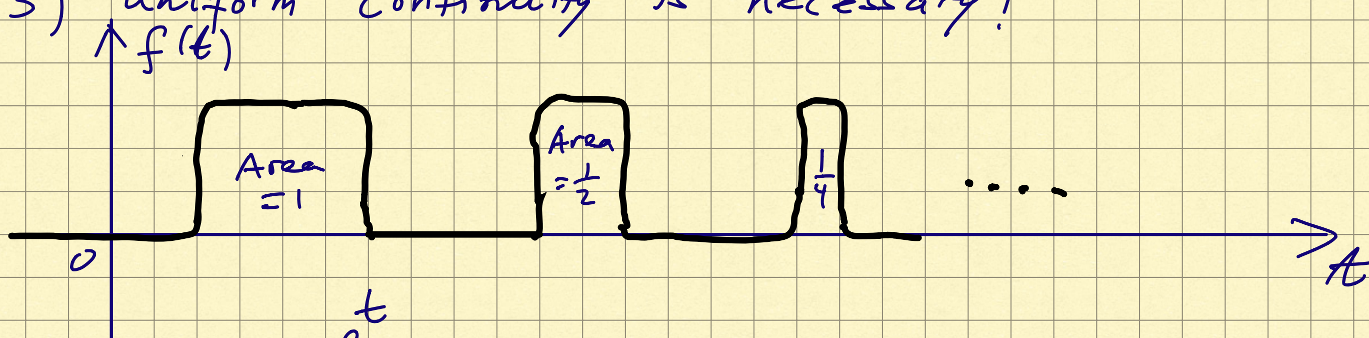
δ will depend on both ε and t;

f is uniformly continuous if $\forall \varepsilon>0$ $\exists \delta = \delta(\varepsilon) > 0$ s.t. $|t'-t|<\delta \Rightarrow |f(t')-f(t)|<\varepsilon$.

δ depends only on ε!

2) $f(t)$ can take negative values the result still holds. (Khalil's book has the proof)

3) uniform continuity is necessary!



$$\lim_{t\to\infty}\int_0^t f(s)\,ds = 1+\frac{1}{2}+\frac{1}{4}+\cdots<\infty$$

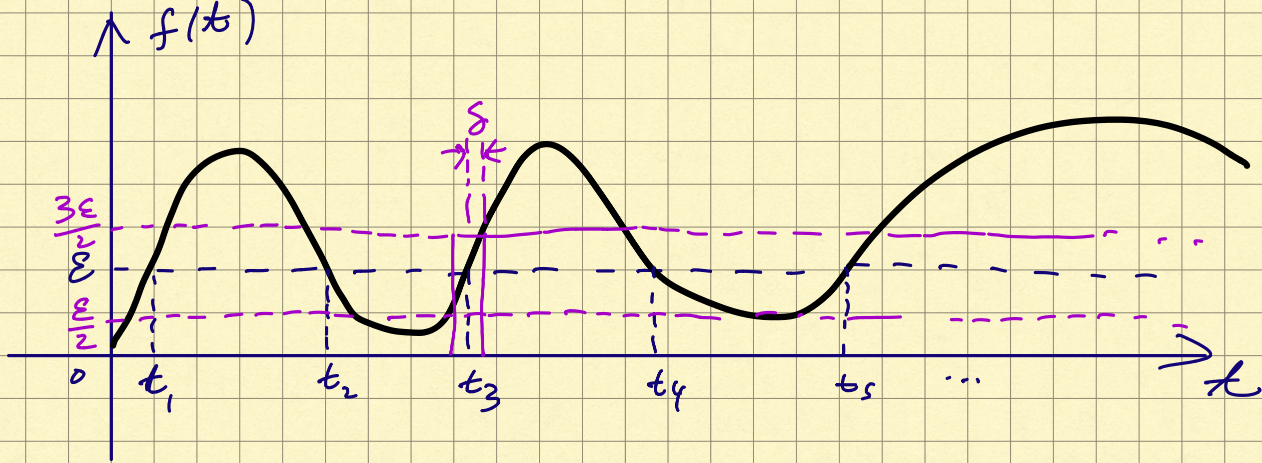
$f(t)\not\to 0$ b/c $f'(t)$ is unbounded

Proof (of Barbalat)

By contradiction.

Assume $f(t) \not\to 0$ as $t \to \infty$.

Then $\exists\, \varepsilon > 0$ and a sequence $\{t_k\}$ s.t. $t_k \geq 0$ and $t_k \to \infty$ as $k \to \infty$ s.t. $f(t_k) \geq \varepsilon,\ \forall k$.



By uniform continuity, $\exists\, \delta > 0$ s.t.

$$|t - t_k| < \delta \quad \Rightarrow \quad |f(t_k) - f(t)| < \varepsilon/2, \ \forall k.$$

Then, for all $k \geq 0$ and all $t \in [t_k, t_k + \delta)$

$$f(t) = \underbrace{f(t_k)}_{\geq \varepsilon} + \underbrace{f(t) - f(t_k)}_{> -\varepsilon/2}$$

$$> \varepsilon/2, \qquad \forall k \geq 0$$

$|f(t_k) - f(t)| < \varepsilon/2$ for $|t - t_k| < \delta$

$$\Rightarrow \int_0^\infty f(t)\,dt \geq \sum_{k=0}^{\infty} \underbrace{\int_{t_k}^{t_k+\delta} f(t)\,dt}_{> \frac{\varepsilon}{2} \cdot \delta}$$

cannot exist.

∎

Corollary Let $x(t)$, $\dot{x}(t)$ be bdd, W continuous, and

$$\int_0^\infty W(x(t))\,dt$$

exists and is finite. Then $W(x(t)) \to 0$ as $t \to \infty$.

Back to proof of main thm.:

$$\int_0^\infty W(x(t))\,dt < \infty$$

where $x(t)$ is a bdd trajectory of $\dot{x} = f(x)$.

Since f is continuous and $x(t)$ is bdd, $f(x(t))$ is bdd $\Rightarrow \dot{x}(t)$ is bdd.

So, $W(x(t)) \longrightarrow 0$ as $t\to\infty$ by Barbalat. $\blacksquare$

Back to example:

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} (\theta - \hat{\theta} - 1)x \\ x^2 \end{pmatrix}$$

$$V(x, \hat{\theta}) = \tfrac{1}{2}x^2 + \tfrac{1}{2}(\hat{\theta} - \theta)^2$$

$$\dot{V}(x, \hat{\theta}) = -x^2$$

$$W(x, \hat{\theta}) = x^2$$

$\int_0^\infty x^2(t)\,dt$ exists and is finite:

$$0 \le \int_0^t x^2(s)\,ds \le V(x(0)) < \infty, \quad \forall t \ge 0$$

$W(x, \hat{\theta})$ is continuous

$x(t)$, $\hat{\theta}(t)$, $\dot{x}(t)$, $\dot{\hat{\theta}}(t)$ are bdd.:

$$\underbrace{\tfrac{1}{2}x^2(t) + \tfrac{1}{2}(\hat{\theta}(t) - \theta)^2}_{V(x(t), \hat{\theta}(t))} \le V(x(0), \hat{\theta}(0)) < \infty \quad \forall t \ge 0$$

$\Rightarrow$ $x(t)$, $\hat{\theta}(t)$ bdd

$|\dot{x}(t)| = |(\theta - \hat{\theta}(t) - 1)x(t)|$ bdd

$|\dot{\hat{\theta}}(t)| = \dot{\hat{\theta}}(t) = x^2(t)$ bdd

By weak Lyapunov criterion, $W(x(t)) = x^2(t) \to 0$

$\Rightarrow$ $x(t) \to 0$ as $t \to \infty$.

On the other hand, can't say anything about convergence of $\hat{\theta}(t)$!

Why? LaSalle: $(x(t), \hat{\theta}(t)) \xrightarrow{t \to \infty} \{(0, \hat{\theta}) : \hat{\theta} \in \mathbb{R}\}$

Preview:

- $V(x) = \frac{1}{2}x^2$ is actually useful to show $x(t) \to 0$ as $t \to \infty$.
- connection to observability (ECE 515 review: Observability Gramians etc.)