Adaptive Control

Adaptation: dynamic process by which the controller adjusts its interaction with a system in order to carry out an objective (or reach a goal) w/o exact knowledge of the system.

Some (incomplete) history:

1950s: gain scheduling
early Model Reference Adaptive Control (MRAC)

1958: R. Kalman — self-tuning controller (regulator) for the linear quadratic problem.

1960s: stability of adaptive controllers
Lyapunov stability
adaptation = learning (Feldbaum, Tsypkin)

1966: Parks — Lyapunov redesign approach to MRAC

1970s: stability analysis (Narendra, Morse, ...)

1980s: limitations (Rohrs et al.: sensitivity to unmodeled dynamics)

1983: Morse's conjecture $\dot{x} = ax + bu$
$a \in \mathbb{R}$
$b \neq 0$

cannot stabilize w/o knowledge of sign(b)

1983: Nussbaum disproves Morse's conjecture
1984: Willems - Byrnes: simplified Nussbaum's construction
exploration vs. exploitation

Example: adaptive regulation

$\dot{x} = \theta x + u$ $\qquad$ $x(t) \in \mathbb{R}$ scalar state
$\qquad\qquad\qquad\quad$ $u(t) \in \mathbb{R}$ —..— input

$\theta \in \mathbb{R}$ is unknown

Goal: output regulation

$x(t) \longrightarrow 0$ as $t \to \infty$
$u(t)$ bounded

What if θ is known?

- $\theta < 0$ $\qquad$ $u \equiv 0$ is optimal

$u \equiv 0$: $\quad x(t) = x(0) e^{\theta t} \longrightarrow 0$ (since $\theta < 0$)

- $\theta \equiv 0$ $\qquad$ $u = -x$

$\dot{x} = -x$ $\qquad$ $x(t) = x(0) e^{-t} \longrightarrow 0$

- $\theta > 0$ $\qquad$ $u = -(\theta + 1)x$

$\dot{x} = \theta x - (\theta + 1)x \iff \dot{x} = -x$

What if θ is unknown?

Adaptive control law:

$\dot{x} = \theta x + u$ $\qquad$ $u = -(\hat{\theta} + 1)x$

$\hookrightarrow$ not an estimate of θ!

$\boxed{\dot{\hat{\theta}} = x^2}$

– tuning law for the controller

Closed-loop system (plant + controller):

$$\left.\begin{aligned} \dot{x} &= \theta x - (\hat{\theta} + 1)x \\ \dot{\hat{\theta}} &= x^2 \end{aligned}\right\}$$

do we achieve stability (regulation)?

$x(t) \longrightarrow 0$ as $t \to \infty$

Note: • the closed-loop system is nonlinear even though the unknown plant is linear

• the controller is dynamic (i.e., it has a state)

$$\hat{\theta}(t) = \hat{\theta}(0) + \int_0^t x^2(s)\,ds$$

We will use Lyapunov stability theory.

$V(x) = \frac{1}{2}x^2$ (first attempt)

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} (\theta - \hat{\theta} - 1)x \\ x^2 \end{pmatrix}$$

$$\dot{V} \equiv \frac{d}{dt} V(x(t))$$

$$= \frac{\partial V}{\partial x}(x(t))\,\dot{x}(t)$$

$$= x(t)\,(\theta - \hat{\theta}(t) - 1)\,x(t)$$

unknown but fixed

$$\dot{V} = \underbrace{(\theta - \hat{\theta} - 1)}_{\text{can be} > 0} x^2$$

$V(x) = \frac{1}{2}x^2$ may not be the best choice

Try: $V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\hat{\theta} - \theta)^2$

– depends on both of the state variables, x and $\hat{\theta}$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial \hat{\theta}}\dot{\hat{\theta}}$$

$$= x\dot{x} + (\hat{\theta} - \theta)\dot{\hat{\theta}}$$

$$= x(\theta - \hat{\theta} - 1)x \quad + \quad (\hat{\theta} - \theta)x^2$$

$$= -x^2 + \cancel{(\theta - \hat{\theta})x^2} + \cancel{(\hat{\theta} - \theta)x^2}$$

$$\boxed{= -x^2} \quad \leq 0$$

Is this enough to give $x(t) \longrightarrow 0$ as $t \to \infty$?

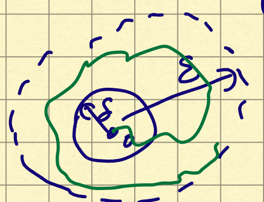
Review: Lyapunov Stability

$$\dot{x} = f(x) \qquad f(0) = 0 \qquad (x \equiv 0 \text{ is an equilibrium})$$

Is $x = 0$ a stable equilibrium?

Stability: $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$$|x(0)| < \delta \implies |x(t)| < \varepsilon, \quad \forall t \geq 0$$



Asymptotic stability: stability + attractivity

$$\exists \delta' > 0 \text{ s.t. } |x(0)| < \delta' \implies \lim_{t \to \infty} x(t) = 0$$

Global Asymptotic Stability (GAS)

$\forall \varepsilon > 0, \ \forall x(0) \qquad |x(t)| < \varepsilon$ after some t_0

+ asymptotic stability

$$\dot{x} = f(x)$$

Thm (Lyapunov stability criterion)

Let V be a C^1 and positive definite fcn of state:

$V(x) \geq 0 \quad \forall x$, and $V(x) = 0$ iff $x = 0$.

Define its time-derivative along $\dot{x} = f(x)$ by

$$\dot{V}(x) = \nabla V(x)^T f(x).$$

Then:

- if $\dot{V}(x) \leq 0$ everywhere, then 0 is a stable equilibrium.
- if $\dot{V}(x) < 0$ for all $x \neq 0$ and $\dot{V}(0) = 0$, then 0 is AS.
- if $\dot{V}(x) < 0$ for all $x \neq 0$ and $\dot{V}(0) = 0$, and V is radially unbounded ($V(x) \to \infty$ as $|x| \to \infty$), then 0 is GAS.

Back to our example:

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} (\theta - \hat{\theta} - 1)x \\ x^2 \end{pmatrix} \qquad (x, \hat{\theta}) = (0, \theta)$$

$$V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\hat{\theta} - \theta)^2 \qquad C^1, \ p.d.$$

$$\dot{V} = -x^2$$

- system is stable (in the sense of Lyapunov)
- not enough info to get AS

$$\dot{V}(x, \hat{\theta}) = -x^2$$

$$\dot{V}(x, \hat{\theta}) = 0 \quad \text{for all } (x, \hat{\theta}) \text{ with } x = 0$$

Need better tools : weak Lyapunov fcn

Key takeaways:

1) closed-loop dynamics of plant + adaptive controller are generally nonlinear (even if the plant is linear)

2) the controller is dynamic (has a nontrivial state dynamics)
   — this enables learning

3) Standard Lyapunov stability theory not strong enough; need to account for $\hat{\theta}$

[$\dot{V}(x, \tilde{\theta}) = -x^2$, indep. of $\hat{\theta}$]

4) $\hat{\theta}$ need <u>not</u> converge to θ!