Problems to be handed in

1 Consider a smooth control-affine system

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $f(0) = 0$. Assume that (1) admits a control Lyapunov function V, and define

$$a(x) := L_f V(x), \qquad b(x) := \left(L_{g_1} V(x), \ldots, L_{g_m} V(x)\right)^\mathsf{T}.$$

With these definitions, we have shown that the state feedback law

$$k(x) := \begin{cases} -\frac{a(x)+\sqrt{a^2(x)+|b(x)|^4}}{|b(x)|^2} b(x), & b(x) \neq 0 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

is a continuous stabilizing control for (1). In this problem, we will obtain (2) as a solution to an infinite-horizon optimal stabilizing control problem. Fix some $x \neq 0$ and consider the following linear system with a one-dimensional state z and an m-dimensional input v:

$$\dot{z} = a(x)z + b(x)^\mathsf{T} v.$$

For this system, consider the infinite-horizon linear quadratic control problem of minimizing

$$\int_0^\infty \left(|v(t)|^2 + b^2(x) z^2(t)\right) \mathrm{d}t$$

over all stabilizing controls $v : [0, \infty) \to \mathbb{R}^m$. Show that the solution of this problem gives rise to (2).

Note: Don't forget the case when $b(x) = 0$!

2 Consider the SISO system $\dot{x} = xu$. We wish to minimize

$$\int_0^\infty \left(\frac{1}{8}x^4(t) + \frac{1}{2}u^2(t)\right) \mathrm{d}t$$

over all controls $u(\cdot)$ yielding $x(t) \to 0$ as $t \to \infty$. Show that an optimal control can be given in state feedback form $k(x) = -cx^2$ for some $c > 0$.

Hint: Try a quadratic candidate Bellman–Lyapunov function $V(x) = px^2$ and find a suitable p by solving the HJB equation.

3 For the SISO system $\dot{x} = x^2 + u$, find the value function $V(x)$ and the optimal feedback control $k(x)$ for the problem of minimizing

$$\int_0^\infty \left(x^2(t) + u^2(t)\right) \mathrm{d}t.$$