Problems to be handed in

1 Consider the autonomous dynamical system

$$\dot{x} = f(x) \tag{1}$$

where $x(t) \in \mathbb{R}^n$. Let a vector $\xi \in \mathbb{R}^n$ be given. Then, for $t \geq s \geq 0$, let $\varphi_{s,t}(\xi)$ denote the point $x(t)$ on the trajectory of this system starting from $x(s) = \xi$, or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{s,t}(\xi) = f(\varphi_{s,t}(\xi)), \qquad t \geq s$$

with the initial condition $\varphi_{s,s}(\xi) = \xi$. By time invariance, $\varphi_{s,t}(\xi) = \varphi_{0,t-s}(\xi)$. We say that the system (1) is *exponentially contracting* if there exist constants $c, \lambda > 0$ such that

$$|\varphi_{s,t}(\xi) - \varphi_{s,t}(\xi')| \leq ce^{-\lambda(t-s)}|\xi - \xi'|$$

for all $t \geq s \geq 0$ and all $\xi, \xi' \in \mathbb{R}^n$. In other words, an exponentially contracting system "forgets" its initial condition exponentially fast.

Now, let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function which is m-strongly convex, i.e.,

$$F(y) \geq F(x) + \nabla F(x)^\top(y - x) + \frac{m}{2}|y - x|^2, \qquad \forall x, y \in \mathbb{R}^n$$

Prove that the gradient flow $\dot{x} = -\nabla F(x)$ is exponentially contracting.

2 Consider the problem of estimating the scalar parameter θ from online observations $(u(t), y(t))$ related via $y(t) = \theta u(t)$. In class, we have discussed the gradient method

$$\dot{\widehat{\theta}} = -\gamma \nabla J_t(\widehat{\theta}),$$

where $J_t(\widehat{\theta}) := \frac{1}{2}(\widehat{\theta}u(t) - y(t))^2$ is the instantaneous cost at time t and $\gamma > 0$ is a fixed adaptation gain. We have shown that the parameter estimation error $\tilde{\theta}(t) := \widehat{\theta}(t) - \theta$ evolves according to the ODE

$$\dot{\tilde{\theta}} = -\gamma u^2(t)\tilde{\theta}.$$

(a) Prove the above equation for $\tilde{\theta}$ has the solution

$$\tilde{\theta}(t) = \exp\left(-\gamma \int_0^t u^2(s)\,\mathrm{d}s\right)\tilde{\theta}(0).$$

(b) We say that $\tilde{\theta}$ is *Uniformly Exponentially Convergent* (UEC) if there exist some $c, \lambda > 0$ such that

$$|\tilde{\theta}(t)| \leq ce^{-\lambda(t-t_0)}|\tilde{\theta}(t_0)|, \qquad \forall t \geq t_0 \geq 0.$$

Prove that $\tilde{\theta}$ is UEC if and only if the input u has the persistent excitation (PE) property.

3 Recall that the minimizers

$$\hat{\theta}(t) = \arg\min_{\hat{\theta}} \left\{ \frac{1}{2} \int_0^t (\hat{\theta}^\top \bar{\phi}(s) - \bar{z}(s))^2 \, \mathrm{d}s + \frac{1}{2} (\hat{\theta} - \hat{\theta}_0)^\top Q_0 (\hat{\theta} - \hat{\theta}_0) \right\}, \qquad t \geq 0$$

for a given initial condition $\hat{\theta}(0) = \hat{\theta}_0$ and a fixed symmetric and positive definite matrix Q_0 can be generated by the flow

$$\begin{aligned} \dot{\hat{\theta}} &= -P \bar{e} \bar{\phi} \\ \dot{P} &= -P \bar{\phi} \bar{\phi}^\top P \end{aligned}$$

with initial conditions $\hat{\theta}(0) = \hat{\theta}_0$ and $P(0) = Q_0^{-1}$. Here, $\bar{e}(t) := \hat{\theta}(t)^\top \bar{\phi}(t) - \bar{z}(t)$ is the (normalized) output prediction error. Fill in the details in the proof of $\bar{e}, \dot{\hat{\theta}} \in L_2 \cap L_\infty$.

4 Consider the first-order scalar plant

$$\dot{y} = -ay + bu \tag{2}$$

where $a > 0$ (so the system is stable) and $b \neq 0$.

(a) Derive an explicit expression for the output $y(t)$ of (2) due to the sinusoidal input $u(t) = \sin \omega t$, where $\omega \neq 0$ is a constant frequency.

(b) Using the result of part (a), show that the steady-state output of (2) is given by

$$y_{\mathrm{ss}}(t) = A \sin(\omega t + \alpha),$$

and give explicit expressions for the amplitude A and the phase α in terms of the plant parameters a, b and the input frequency ω.

(c) Show that the sinusoidal input $u(t) = \sin \omega t$ is sufficiently rich in the sense that $\phi = (u, -y_{\mathrm{ss}})^\top$ has the PE property.