Problems to be handed in

1 This problem introduces a useful technique for obtaining global estimates for C^1 functions from local information (such as their gradients).

(a) Let a C^1 function $f : \mathbb{R}^n \to \mathbb{R}^n$ be given. Prove that there exists a continuous matrix-valued function $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, such that

$$f(y) = f(x) + G(x, y - x)(y - x)$$

for all $x, y \in \mathbb{R}^n$.

(b) We say that a C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ is *M-smooth* if

$$|\nabla f(x) - \nabla f(y)| \le M|x - y|, \qquad \forall x, y \in \mathbb{R}^n$$

(where $|\cdot|$ denotes the usual Euclidean norm). Prove that if f is M-smooth, then

$$f(y) - f(x) - \nabla f(x)^\top (y - x) \le \frac{M}{2}|x - y|^2.$$

Hint: In both cases, try differentiating the C^1 function $F(t) := f(x + t(y - x))$, $t \in [0, 1]$.

2 Consider a nonlinear system

$$\dot{x} = f(x, u)$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, where $f(\cdot, \cdot)$ satisfies $f(0, 0) = 0$ and is C^1 in both arguments. Suppose that this system admits a control Lyapunov function (CLF) $V_0(x)$ and a C^1 stabilizing state feedback law $u = k_0(x)$ satisfying $k_0(0) = 0$. Use an appropriate extension of the backstepping method to prove that the system

$$\begin{aligned} \dot{x} &= f(x, \xi) \\ \dot{\xi} &= h(x, \xi) + u \end{aligned}$$

with $x \in \mathbb{R}^n$ and $\xi, u \in \mathbb{R}^m$ admits a CLF $V_1(x, \xi)$ and a continuous stabilizing state feedback law $u = k_1(x, \xi)$. Here, $h(\cdot, \cdot)$ is continuous and satisfies $h(0, 0) = 0$.

Hint: Problem 1(a) may be handy.

3 Consider the linear system

$$\begin{aligned} \dot{x} &= \theta x + \xi_1 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= u \end{aligned}$$

with $x, \xi_1, \xi_2, u \in \mathbb{R}$. The parameter $\theta \in \mathbb{R}$ is unknown. In class, we have used adaptive integrator backstepping to construct a CLF and an adaptive stabilizing controller for the corresponding system without the second integrator, i.e., when $\xi_1 = u$. Iterate on that construction to construct a CLF and an adaptive stabilizing controller for the above system with two integrators.