Problems to be handed in

1. This problem introduces a useful technique for obtaining global estimates for \(C^1\) functions from local information (such as their gradients).

   (a) Let a \(C^1\) function \(f : \mathbb{R}^n \rightarrow \mathbb{R}^n\) be given. Prove that there exists a continuous matrix-valued function \(G : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}\), such that
   \[f(y) = f(x) + G(x, y - x)(y - x)\]
   for all \(x, y \in \mathbb{R}^n\).

   (b) We say that a \(C^1\) function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) is \(M\)-smooth if
   \[|\nabla f(x) - \nabla f(y)| \leq M|x - y|, \quad \forall x, y \in \mathbb{R}^n\]
   (where \(\cdot, \cdot\) denotes the usual Euclidean norm). Prove that if \(f\) is \(M\)-smooth, then
   \[f(y) - f(x) - \nabla f(x)^T(y - x) \leq \frac{M}{2}|x - y|^2.\]

   **Hint:** In both cases, try differentiating the \(C^1\) function \(F(t) := f(x + t(y - x)), t \in [0, 1]\).

2. Consider a nonlinear system
   \[
   \dot{x} = f(x, u)
   \]
   with \(x \in \mathbb{R}^n, u \in \mathbb{R}^m\), where \(f(\cdot, \cdot)\) satisfies \(f(0, 0) = 0\) and is \(C^1\) in both arguments. Suppose that this system admits a control Lyapunov function (CLF) \(V_0(x)\) and a \(C^1\) stabilizing state feedback law \(u = k_0(x)\) satisfying \(k_0(0) = 0\). Use an appropriate extension of the backstepping method to prove that the system
   \[
   \dot{x} = f(x, \xi) \\
   \dot{\xi} = h(x, \xi) + u
   \]
   with \(x \in \mathbb{R}^n\) and \(\xi, u \in \mathbb{R}^m\) admits a CLF \(V_1(x, \xi)\) and a continuous stabilizing state feedback law \(u = k_1(x, \xi)\). Here, \(h(\cdot, \cdot)\) is continuous and satisfies \(h(0, 0) = 0\).

   **Hint:** Problem 1(a) may be handy.

3. Consider the linear system
   \[
   \dot{x} = \theta x + \xi_1 \\
   \dot{\xi}_1 = \xi_2 \\
   \dot{\xi}_2 = u
   \]
   with \(x, \xi_1, \xi_2, u \in \mathbb{R}\). The parameter \(\theta \in \mathbb{R}\) is unknown. In class, we have used adaptive integrator backstepping to construct a CLF and an adaptive stabilizing controller for the corresponding system without the second integrator, i.e., when \(\xi_1 = u\). Iterate on that construction to construct a CLF and an adaptive stabilizing controller for the above system with two integrators.