Technical Notes and Correspondence

Note to the Reader: The following paper has spurred much discussion in the adaptive controls community for a number of years now. Because of the subsequent research it generated, a commentary was felt to be appropriate. Hence, at the end of this paper, read on for some insight provided by Karl Åström.

RAY DECARLO

Robustness of Continuous-Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics

CHARLES E. ROHRS, LENA VALAVANI, MICHAEL ATHANS, AND GUNTER STEIN

***Abstract*—This paper examines the robustness properties of existing adaptive control algorithms to unmodeled plant high-frequency dynamics and unmeasurable output disturbances. It is demonstrated that there exist two infinite-gain operators in the nonlinear dynamic system which determines the time-evolution of output and parameter errors. The pragmatic implication of the existence of such infinite-gain operators is that 1) sinusoidal reference inputs at specific frequencies and/or 2) sinusoidal output disturbances at any frequency (including dc), can cause the loop gain to increase without bound, thereby exciting the unmodeled high-frequency dynamics, and yielding an unstable control system. Hence, it is concluded that existing adaptive control algorithms as they are presented in the literature referenced in this paper, cannot be used with confidence in practical designs where the plant contains unmodeled dynamics because instability is likely to result. Further understanding is required to ascertain how the currently implemented adaptive systems differ from the theoretical systems studied here and how further theoretical development can improve the robustness of adaptive controllers.**

I. INTRODUCTION

This paper reports the outcome of an investigation of the stability and robustness properties of a wide class of adaptive control algorithms in the presence of unmodeled high-frequency dynamics and persistent unmeasurable output disturbances. Every physical system has such (parasitic) high-frequency dynamics; in nonadaptive designs these limit the crossover frequency and require the control system to contain adequate gain and phase margins. Also, every control system must be able to operate in the presence of unmeasured and possibly persistent disturbances without going unstable. In nonadaptive linear control designs the presence of disturbances does not impact upon the closed-loop stability issue.

It should be stressed that the existing adaptive control algorithms, when used to control an unknown linear time-invariant plant, result in a highly nonlinear and time-varying closed-loop system whose stability does depend upon the external inputs (reference inputs and disturbance inputs). Hence, it is important to analyze the stability of the adaptive closed-loop design and to inquire about its global stability properties. This has provided the motivation for the research reported in this paper.

Due to space limitations we cannot possibly provide in this paper analytical and simulation evidence of all conclusions outlined in the Abstract. Rather, we summarize the basic approach only for a single class of continuous-time algorithms that includes those of Monopoli [14], Narendra and Valavani [1], and Feuer and Morse [2]. However, the same analysis techniques have been used to analyze more complex classes of (1) continuous-time adaptive control algorithms due to Narendra, Lin, and Valavani [3], both algorithms suggested by Morse [4], and the algorithms suggested by Egardt [7] which include those of Landau and Silveira [6], and Kreisselmeier [19]: and (2) discrete-time adaptive control algorithms due to Narendra and Lin [22], Goodwin, Ramadge, and Caines [5] (the so-called dead-beat controllers), and those developed in Egardt [17], which include the self-tuning regulator of Åström and Wittenmark [18] and that due to Landau [20]. The thesis by Rohrs [15] contains the full analysis and simulation results for the above classes of existing adaptive algorithms.

The end of the 1970's marked significant progress in the theory of adaptive control, both in terms of obtaining global asymptotic stability proofs [1]–[7] as well as in unifying diverse adaptive algorithms, the derivation of which was based on different philosophical viewpoints [8], [9].

Unfortunately, the stability proofs of all these algorithms have in common a very restrictive assumption. For continuous-time implementation this assumption is that the number of poles minus the number of zeros of the plant, i.e., its relative degree, is known. The counterpart of this assumption for the discrete-time systems is that the pure delay in the plant is exactly an integer number of sampling periods and that this integer is known. It is also assumed that an upper bound for the number of poles in the plant is known in both the continuous-time and discrete-time formulation. Finally, global parameter convergence requires that the inputs satisfy a "sufficiently rich" condition.

The restrictive relative degree assumption, in turn, is equivalent to enabling the designer to realize for an adaptive algorithm, a positive real error transfer function, on which all stability proofs have heavily hinged to-date [8]. Positive realness implies that the phase of the system cannot exceed $\pm 90°$ for all frequencies, while it is a well-known fact that models of physical systems become very inaccurate in describing actual plant high-frequency phase characteristics. Moreover, for practical reasons, most controller designs need to be based on models which do not contain all of the plant dynamics, in order to keep the complexity of the required adaptive compensator within bounds.

Motivated from such considerations, researchers in the field in the early 1980's began investigating the robustness of adaptive algorithms to violation of the restrictive (and unrealistic) assumption of knowledge of the plant order and its relative degree. Ioannou and Kokotovic [10] obtained error bounds for adaptive observers and identifiers in the presence of unmodeled dynamics, while such analytical results were harder to obtain for reduced-order adaptive controllers. The first such result, obtained by Rohrs *et al.* [11], consists of "linearization" of the error equations, under the assumption that the overall system is in its final approach to convergence. Ioannou and Kokotovic [12] later obtained local stability results in the presence of unmodeled dynamics, and showed that the speed ratio of slow versus fast (unmodeled) dynamics directly affected the stability region. Even the local stability results of [12] can only be attained when the speed ratio is small, i.e., when the unmodeled dynamics are much faster than the modeled part of the process. Earlier simulation studies by Rohrs *et al.* [13] had already shown the high sensitivity of adaptive algorithms to disturbances and unmodeled dynamics, generation of high-frequency control inputs and ultimately instability. Simple root-locus type plots for the linearized system in [11] showed how the presence of unmodeled dynamics could bring about instability of the overall

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system. It was also shown there that the generated frequencies in the adaptive loop depend nonlinearly on the magnitudes of the reference input and output. Kosut and Friedlander [23] have also tried to define the class of plant perturbations under which adaptive controllers will retain stability. They have also found severe limitations as in [11], on the reference input and output characteristics for which the linearized error equations are stable.

The main contribution of this paper is in showing that two operators inherently present in all algorithms considered (as part of the adaptation mechanism) have infinite gain. As a result, two possible mechanisms of instability are isolated and discussed. It is argued that the destabilizing effects in the presence of unmodeled dynamics can be attributed to either phase (in the case of high-frequency inputs), or, primarily gain considerations (in the case of unmeasurable output disturbances of any frequency, including dc), which result in nonzero steady-state errors. The latter fact is most disconcerting for the performance of adaptive algorithms since it cannot be easily dealt with by additional filtering, given that a persistent disturbance of any frequency can have a destabilizing effect.

Our conclusion is that the adaptive algorithms as published in the literature are likely to produce unstable control systems if they are implemented on physical systems directly as they appear in the literature. The conclusions stem from the results of this paper which show unstable behavior of adaptive systems when these systems are confronted with two premises that cannot be ignored in any physical control design: 1) there are always unmodeled dynamics at sufficiently high frequencies (and it is futile to try to model these dynamics); and 2) the plant cannot be isolated from unknown disturbances (e.g., 60 Hz hum) even though these may be small.

The original version of this paper was presented as [24]. Since that time, there has been a great deal of research generated on the robustness issues raised here. In particular, it has come to appear that some problems presented in this paper may possibly be overcome by sufficient excitation. The following idea of the use of sufficient excitation can be traced back at least to [25]. By using sufficient excitation the nominal adaptive system can be made exponentially stable. Since the system is exponentially stable there is *some* modeling error and *some* disturbances for which stability will be maintained. It should be noted, however, that the amount of modeling error or the amount of disturbance for which the adaptive system can maintain stability may be extremely small.

The issue of sufficient excitation is not considered here. It is a complex issue which will produce many papers of its own. However, the following results pertaining to sufficient excitation should be noted. Since the inputs which cause destabilization in this paper actually provide the usual measure of sufficient excitation it is clear from these results that not just sufficient excitation is needed by sufficient low-frequency excitation. Ioannou and Kokotovic [10] used the term "dominantly rich excitation" in their investigation of the concept. Since the publication of [24], Krause *et al.* [26] have analyzed the infinite gain operator more deeply and extended the analysis of this paper in a significant way to include the effects of the richness of input in the stability analysis. The results of the analysis of [26] show that the input must not only be sufficiently exciting to produce parameter convergence in the nominal system, but that the input must be dominantly rich enough to overcome the destabilizing effects discussed here. Thus, sufficient excitation should not be viewed as a panacea which creates robust adaptive controllers but as a stabilizing effect in a delicate engineering tradeoff.

In Section II the infinite gain of the operators generic to the adaptation mechanism is displayed. Section III contains the development of two possible mechanisms for instability that arise as a result of the infinite gain operators. Simulation results that show the validity of the heuristic arguments in Section III are presented in Section IV. Section V contains the conclusions.

II. The Error Model Structure for a Representation Adaptive Algorithm

The simplest prototype for a model reference adaptive control algorithm in continuous-time has its origins to at least as far back as 1974, in the paper by Monopoli [14]. This algorithm has been proved asymptotically stable only for the case when the relative degree of the plant is unity or at most two. The algorithms published by Narendra and Valavani [1] and Feuer and Morse [2] reduce to the same algorithm for the pertinent case. This algorithm will henceforth be referred to as CA1 (continuous-time algorithm No. 1).

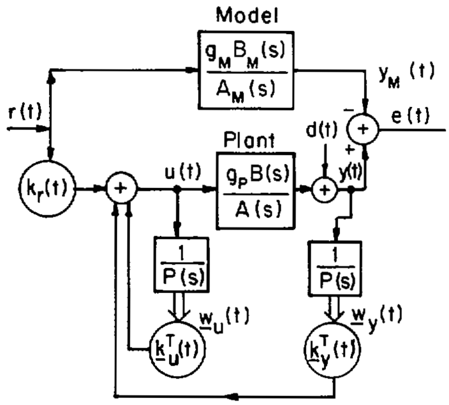

Fig. 1. Controller structure of CA1 with additive output disturbance $d(t)$.

The following equations summarize the dynamical equations that describe it; see also Fig. 1. The equations presented here pertain to the case where a unity relative degree has been assumed. In the equations below, $r(t)$ is the (command) reference input and the disturbance $d(t)$ in Fig. 1 is equal to zero.

plant:
$$y(t) = \frac{g_p B(s)}{A(s)}[u(t)] \tag{1}$$

auxiliary variables:
$$w_{ui}(t) = \frac{s^{i-1}}{P(s)}[u(t)]; \qquad i = 1, 2, \cdots, n-1 \tag{2}$$

$$w_{yi}(t) = \frac{s^{i-1}}{P(s)}[y(t)]; \qquad i = 1, 2, \cdots, n \tag{3}$$

$$w(t) \triangleq \begin{bmatrix} r(t) \\ w_u(t) \\ w_y(t) \end{bmatrix}; \; k(t) \triangleq \begin{bmatrix} k_r(t) \\ k_u(t) \\ k_y(t) \end{bmatrix} \tag{3a}$$

model:
$$y_M(t) = \frac{g_M B_M(s)}{A_M(s)}[r(t)] \tag{4}$$

control input:
$$u(t) = k^T(t)w(t) = k^{*T}(t)w(t) + \tilde{k}^T(t)w(t)$$
$$= k^{*T}(t)w(t) + \tilde{u}(t) \tag{5}$$

output error:
$$e(t) = y(t) - y_M(t) \tag{6}$$

parameter adjustment law:
$$\dot{k}(t) = \dot{\tilde{k}}(t) = \Gamma w(t)e(t); \qquad \Gamma = \Gamma' > O \tag{7}$$

nominal controlled plant:
$$\frac{g^*B^*(s)}{A^*(s)} = \frac{k_r^* g_p B(s)P(s)}{A(s)P(s) - A(s)K_u^*(s) - g_p B(s)K_y^*(s)} \tag{8}$$

error equation:
$$e(t) = \left(\frac{g^*B^*(s)}{A^*(s)} - \frac{g_M B_M(s)}{A_M(s)}\right)[r(t)]$$
$$+ \frac{g^*B^*(s)}{A^*(s)}\left[\frac{\tilde{k}^T(t)w(t)}{k_r^*}\right]. \tag{9}$$

In the above equations the following definitions apply:

$$k(t) \triangleq k^* + \tilde{k}(t) \tag{10}$$

where k^* is a constant $2n$ vector.

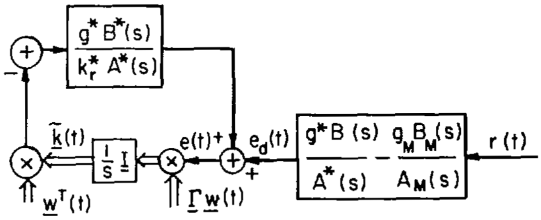

Fig. 2. Error system for CA1.

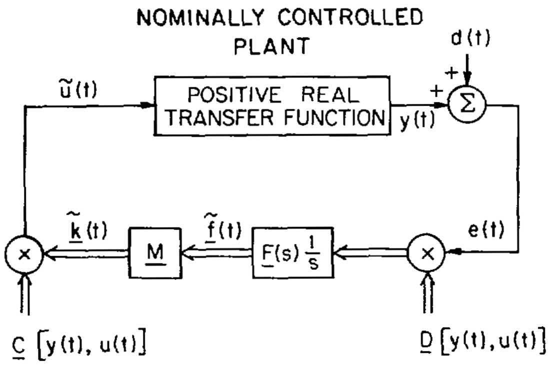

Fig. 3. Block diagram of a generic adaptive controller.

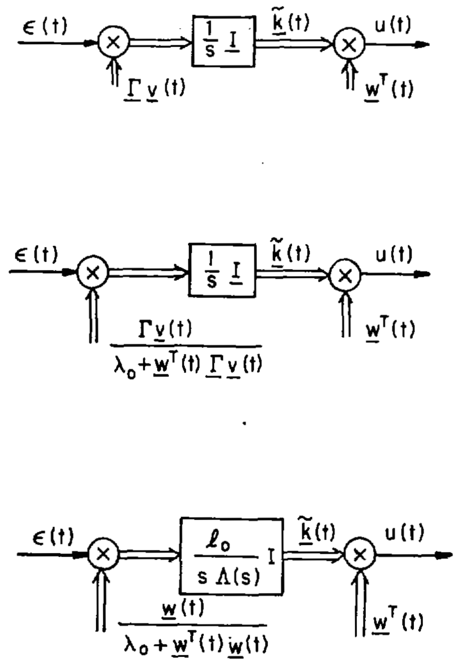

Fig. 4. The infinite gain operators of CA2, CA3, and CA4.

$$K_u^*(s) \triangleq k_{u(n-1)}^* s^{n-2} + k_{u(n-2)}^* s^{n-3} + \cdots + k_{u1}^*$$

where k_{ui}^* is the ith component of $\boldsymbol{k}_u^*$.

$$K_y^*(s) \triangleq k_{yn}^* s^{n-1} + k_{y(n-1)}^* s^{n-2} + \cdots + k_{y1}^*$$

where k_{yi}^* is the ith component of $\boldsymbol{k}_y^*$. In the preceding equations we have tried to preserve the conventional literature notation [3]–[5], with $P(s)$ representing the characteristic polynomial for the state variable filters and $\tilde{k}(t)$ the parameter misalignment vector. In (8) $g^*B^*(s)/A^*(s)$ represents the closed-loop plant transfer function that would result if $\tilde{k}$ were identically zero, i.e., if a constant control law $\boldsymbol{k} = \boldsymbol{k}^*$ were used. Under the conventional assumption that the plant relative degree is exactly known and, if $B_M(s)$ divides $P(s)$, then $\boldsymbol{k}^*$ can be chosen [1], such that

$$\frac{g^*B^*(s)}{A^*(s)} = \frac{g_M B_M(s)}{A_M(s)}. \tag{11}$$

If the relative degree assumption is violated, $g^*B^*(s)/A^*(s)$ can only get as close to $g_M B_M(s)/A_M(s)$ as the feedback structure of the controller allows. The first term on the right-hand side of (9) results from such a consideration. Note that if (11) were satisfied, (9) reduces to the familiar error equation form that has appeared in the literature [8] for exact modeling. For more details the reader is referred to the literature cited in this section as well as to [15].

Fig. 2 represents, in block diagram form, the combination of parameter adjustment law and error equations described by (7) and (9).

In general, existing continuous-time algorithms [1]–[9] can be classified into four groups labeled here and in [15] as CA1, CA2, CA3, CA4, respectively, for continuous-time algorithms 1, 2, 3, 4. Fig. 3 represents a generalized error structure which can be particularized to describe the error loop of any one of the existing adaptive algorithms, both in continuous time and (by its discrete analog) in discrete time as well. In Fig. 3, the forward loop consists of a positive real transfer function, while the feedback path comprises the adaptation mechanism (parameter adjustment), which contains the infinite gain operator(s). In the figure, the error system input $\tilde{u}(t)$ is synthesized according to

$$\tilde{u}(t) = \tilde{k}^T(t) C[y(t), u(t)]$$

$$f(t) = F(s)\frac{1}{s}[D[y(t), u(t)]e(t)]$$

where

- C is a linear time-invariant system representing an observer or an auxiliary state generator
- M is usually a memoryless map, often the identify map
- D is a linear time-invariant system analogous to C and often identical to it
- $F(s)$ a stable transfer function, often the identity
- $\tilde{k}(t)$, $\tilde{f}(t)$ parameter error vectors related by $\tilde{k}(t) = M[\tilde{f}(t)]$.

The above-mentioned four classes of adaptive algorithms have in common the error feedback loop structure and the basic ingredients of the parameter update mechanism, i.e., multiplication-integration-multiplication, which forms the feedback part of the loop and is shown [15] to constitute the *infinite gain operators* present in *all* existing adaptive algorithms. These operators are shown in Fig. 4 as they apply specifically to CA2, CA3, CA4. Analogously, the discrete-time algorithms are classified into three distinct groups, DA1, DA2, DA3. The four classes mentioned in the preceding differ in the specific parameterization that realizes the positive real transfer function in the forward path and in the particular details of the parameter adjustment laws [choice of $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{M}$, $\boldsymbol{F}(s)$]. For example, in the simplest class of continuous-time adaptive algorithms (CA1) under perfect modeling, the forward transfer function represents the reference model transfer function, which the controlled plant is able to match exactly, by assumption (11). Unfortunately, when unmodeled dynamics are present, the controlled plant can only match the reference model only up to a certain frequency range and, thus, the forward loop transfer function which represents the nominally controlled plant and determines the error dynamics, loses its positive realness property. This, in conjunction with the infinite gain operator(s) in the feedback loop, can bring about instability.

III. The Infinite Gain Operators

A. Quantitative Proof of Infinite Gain for Operators of CA1

The error system in Fig. 2 consists of a forward linear time-invariant operator representing the nominal controlled plant complete with unmodeled dynamics $g^*B^*/k_r^*A^*$, and a time-varying feedback operator. It is this feedback operator which is of immediate interest. The operator, reproduced in Fig. 5 for the case where w is a scalar and $\Gamma = 1$, is parameterized by the function $w(t)$ and can be represented mathematically as

$$\tilde{u}(t) = G_{w(t)}[e(t)] = \tilde{u}_0 + w(t)\int_0^t w(\tau)e(\tau)\, d\tau. \tag{12}$$

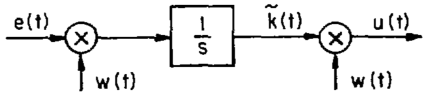

Fig. 5. Infinite gain operator of CA1.

In order to make the notion of the gain of the operator $G_w(t)(\cdot)$ precise, we introduce the following operator theoretic concepts.

Definition 1: A function $f(t)$ from $[0, \infty)$ to R is said to be in L_{2e} if the truncated norm

$$\|f(t)\|^T_{L_2} \triangleq \left(\int_0^T f^2(\tau)\, d\tau \right)^{1/2} \tag{13}$$

is finite for all finite T.

Definition 2: The gain of an operator $G[f(t)]$, which maps functions in L_{2e} into functions in L_{2e} is defined as

$$\|G\| = \sup_{\substack{f(t)\in L_{2e} \\ T\in[0,\infty)}} \frac{\|G[f(t)]\|^T_{L_2}}{\|f(t)\|^T_{L_2}}. \tag{14}$$

If there is no finite number satisfying (14), then G is said to have infinite gain.

Theorem 1: If $w(t)$ is given by

$$w(t) = b + c \sin \omega_0 t \tag{15}$$

for any positive constants b, c, ω_0, the operator of (12) has infinite gain.

Proof: The proof consists of constructing a signal $e(t)$, such that

$$\lim_{T\to\infty} \frac{\|G_w[e(t)]\|^T_{L_2}}{\|e(t)\|^T_{L_2}} \tag{16}$$

is unbounded.

Let $e(t) = a \sin(\omega_0 t + \phi)$ with a an arbitrary positive constant, ϕ an arbitrary phase shift, and ω_0 the same constant as in (15). These signals produce

$$w(t)e(t) = ab(\sin \omega_0 t \cos \phi + \cos \omega_0 \sin \phi) + ac \sin \omega_0 t(\sin \omega_0 t \cos \phi + \cos \omega_0 t \sin \phi) \tag{17}$$

$$\tilde{k}(t) \triangleq \tilde{k}_0 + \int_0^t w(\tau)e(\tau)\, d\tau = \tilde{k}_0 + \frac{1}{2} ac \cos \phi \cdot t + \text{constant terms} + \text{periodic terms} \tag{18}$$

$$\tilde{u}(t) = G_w(t)[e(t)] = \tilde{u}_0 + w(t) \int_0^t w(\tau)e(\tau)\, d\tau = \frac{\cos \phi}{2}(abc + ac^2 \sin \omega_0 t)t + \text{constant terms} + \text{periodic terms}. \tag{19}$$

Next, using standard norm inequalities, we obtain from (19)

$$\|\tilde{u}(t)\|^T_{L_2} > \left\| \frac{1}{2} abct \cos \phi + \frac{1}{2} ac^2 t \sin \omega_0 t \right\|^T_{L_2} - (K_1' T)^{1/2} \tag{20}$$

with K_1' a finite constant. Now

$$\left\| \frac{1}{2} abct \cos \phi + \frac{1}{2} ac^2 t \cos \phi \sin \omega_0 t \right\|^{T^2}_{L_2} \tag{21}$$

$$\geq \left(\frac{a^2b^2c^2 \cos^2 \phi}{12} + \frac{a^2c^4 \cos^2 \phi}{24} \right) T^3 - K_2T^2 + K_1T + K_0 \tag{22}$$

where K_0, K_1, and K_2 are finite constants. Combining inequalities (20) and (21) we arrive at

$$(\|\tilde{u}(t)\|^T_{L_2})^2 \geq \left(\frac{a^2b^2c^2 \cos^2 \phi}{12} + \frac{a^2c^4 \cos^2 \phi}{24} \right) T^3 - K_2T^2 - K_1T - K_0. \tag{23}$$

Also

$$\{\|e(t)\|^T_{L_2}\}^2 = a^2 \int_0^T \sin 2\omega_0 t\, dt \leq a^2 T. \tag{24}$$

Therefore,

$$\frac{\|\tilde{u}(t)\|^T_{L_2}}{\|e(t)\|^T_{L_2}} \geq \frac{\left[\left(\frac{a^2b^2c^2 \cos^2 \phi}{12} + \frac{a^2c^4 \cos^2 \phi}{24} \right) T^3 - K_2T^2 - K_1T - K_0 \right]^{1/2}}{aT^{1/2}} \xrightarrow{T\to\infty} \infty$$

and, therefore, G_w for w as in (15) *has infinite gain.*

In addition to the fact that the operator $G_w(t)$ from $e(t)$ to $\tilde{u}(t)$ has infinite gain, the operator H_w, from $e(t)$ to $\tilde{k}(t)$ in Fig. 5 also has infinite gain. This operator is described by

$$H_{w(t)}[e(t)] = \tilde{k}_0 + \int_0^t w(\tau)e(\tau)\, d\tau. \tag{25}$$

Theorem 2: The operator $H_{w(t)}$ with $w(t)$ given in (15) has infinite gain.

Proof: Choose $e(t) = a \sin \omega_0 t$ as before. Then $\tilde{k}(t) = H_{w(t)}[e(t)]$ is given by (18). Proof of infinite gain for this operator then follows in exactly analogous steps as in Theorem 1 and is, therefore, omitted.

Remark 1: Both operators G_w and H_w will also have infinite gain for vectors $w(t)$, since the operators' infinite gains can arise from any component of the vector $w(t)$.

B. Two Mechanisms of Instability

In this section, we use the algorithm CA1 to introduce and delineate two mechanisms which may cause unstable behavior in the adaptive system CA1, when it is implemented in the presence of unmodeled dynamics and excited by sinusoidal reference inputs or by disturbances. The arguments made for CA1 are also valid for other classes of algorithms mentioned in Section II *mutatis mutandis.* Since the arguments explaining instability are somewhat heuristic in nature, they are verified by simulations. Representative simulation results are given in Section IV.

In order to demonstrate the infinite gain nature of the feedback operator of the error system of CA1 in Section II, it is assumed that a component of $w(t)$ has the form

$$w_i(t) = b + c \sin \omega_0 t \tag{26}$$

and that the error has the form

$$e(t) = a \sin(\omega_0 t + \phi). \tag{27}$$

The arguments of Section II indicate that, if $e(t)$ and a component of $w(t)$ have distinct sinusoids at a common frequency, the operator $G_{w(t)}$ of (12) and the operator $H_{w(t)}$ of (25) will have infinite gains. Two possibilities for $e(t)$, $w(t)$ to have the forms of (26) and (27) are now considered.

Case 1: If the reference input consists of a sinusoid and a constant, e.g.,

$$r(t) = r_1 + r_2 \sin \omega_0 t \tag{28}$$

where r_1 and r_2 are constants, then the plant output $y(t)$ will contain a constant term and a sinusoid at frequency ω_0 with an arbitrary phase shift ϕ. Consequently, through (2), (3), and (3a), all components of the vector $w(t)$ will contain a constant and a sinusoid of frequency ω_0, with a phase shift.

If the controlled plant matches the model at dc but not at the frequency ω_0, the output error

$$e(t) = y(t) - y_M(t) \tag{29}$$

will contain a sinusoid at frequency ω_0. Thus, the conditions for infinite gain in the feedback path of Fig. 2 have been attained.

Case 2: If a sinusoidal disturbance, $d(t)$, at frequency ω_0 enters the plant output as shown in Fig. 1, the sinusoid will appear in $w(t)$ through the following equation, which replaces (3) in the presence of an output disturbance:

$$w_{yi}(t)=\frac{s^{i-1}}{P(s)}[y(t)+d(t)]; \qquad i=1,\ 2,\ \cdots,\ n. \tag{30}$$

The following equation replaces (6) when an output disturbance is present

$$e(t)=y(t)+d(t)-y_M(t). \tag{31}$$

Any sinusoid present in $d(t)$ will also enter $e(t)$ through (31). Thus, the signals $e(t)$ and $w(t)$ will contain sinusoids of the same frequency and the operators $H_{w(t)}$ and $G_{w(t)}$ will display an infinite gain.

1) Instability Due to the Gain of the Operator G_w of (12): The operator G_w of (12) is not only an infinite gain operator but its gain influences the system in such a manner as to allow arguments using linear systems concepts, as outlined below.

Assume, initially, that the error signal is of the form of (27), i.e., a sinusoid at frequency ω_0. Assume also that a component of $w(t)$ is of the form of (26), i.e., a constant plus a sinusoid at the same frequency ω_0 as the input. The output of the infinite gain operator $G_{w(t)}$ of (12), as given by (19), consists of a sinusoid at frequency ω_0 with a gain which increases linearly with time plus other terms at 0 rad/s (i.e., dc) and other harmonics of ω_0, i.e., $\bar{u}(t) = 1/2\ ac^2t \sin \omega_0 t \cos \phi$ + other terms.

The infinite gain operator manifests its large gain by producing at the output *a sinusoid at the same frequency, ω_0, as the input sinusoid but with an amplitude increasing with time.* By concentrating on the signal at frequency ω_0, and viewing the operator $G_{w(t)}$ as a simple time-increasing gain with no phase shift at the frequency ω_0, and very small gain at other frequencies, we will be able to come up with a mechanism for instability of the error system of Fig. 2, where $G_{w(t)}$ consists of the feedback part of the loop. If the forward path $g^*B^*(s)/k_r^*A^*(s)$, of the error loop of Fig. 2, has less than $\pm 180°$ phase shift at the frequency ω_0, and if the gain of $G_{w(t)}$ were indeed small at all other frequencies, then the high gain of $G_{w(t)}$ at ω_0 would not affect the stability of the error loop. If, however, the forward loop, $g^*B^*(s)/k_r^*A^*(s)$ does have 180° phase shift at ω_0, the combination of this phase shift with the sign reversal will produce a positive feedback loop around the operator $G_{w(t)}$, thereby reinforcing the sinusoid at the input of $G_{w(t)}$. The sinusoid will then increase in amplitude linearly with time, as the gain of $G_{w(t)}$ grows, until the *combined gain* of $G_{w(t)}$ and $g^*B^*(s)/k_r^*A^*(s)$ exceeds unity at the frequency ω_0. At this point, the loop itself will become unstable and all signals will grow without bound very quick (as the effects of the unstable loop and continually growing gain of $G_{w(t)}$ compound).

Since the infinite gain of $G_{w(t)}$ can be achieved at any frequency ω_0, if $g^*B^*(s)/k_r^*A^*(s)$ has $\pm 180°$ shift at any frequency, the adaptive system is susceptible to instability from either a reference input or a disturbance.

Thus, the importance of the relative degree assumption, which essentially allows one to assume that $g^*B^*(s)/k_r^*A^*(s)$ is strictly positive real is seen. The stability proof CA1 *hinges* on the assumption that $g^*B^*(s)/k_r^*A^*(s)$ is strictly positive real and that $G_{w(t)}$ is passive, i.e.,

$$\int_0^\infty G_{w(t)}[e(t)]e(t)\ dt \geq 0. \tag{32}$$

Both properties of positive realness and passivity are properties which are independent of the gain of the operator involved. *However, it is always the case that, due to the inevitable unmodeled dynamics, only a bound is known on the gain of the plant at high frequencies.* Therefore, for a large class of unmodeled dynamics with relative degree two or greater, the operator $g^*B/k_r^*A^*$, will have $\pm 180°$ phase shift at some frequency and be susceptible to unstable behavior if subjected to sinusoidal reference inputs and/or disturbances in that frequency range.

2) Instability Due to the Gain of the Operator H_w of (25): In the previous subsection, the situation was examined where the amplitude of the sinusoidal error $e(t)$ grew with time due to a positive feedback mechanism in the error loop. In this subsection, we explore the situation where the sinusoidal error $e(t)$, is not at a frequency where it will grow due to the error system but, rather, when there exist persistent steady-state errors. Such a persistent error could easily arise when an output sinusoidal disturbance $d(t)$ enters as shown in Fig. 1, causing the persistent sinusoid directly on $e(t)$, through (31), and $w(t)$ through (30).

Assume that a component of $w(t)$ contains a sinusoid at frequency ω_0 as in (26) and that $e(t)$ contains a sinusoid of the same frequency. Then the operator $H_{w(t)}$ has infinite gain and the norm of the output signal of this operator, $\tilde{k}(t)$, increases without bound. The signal $\tilde{k}(t)$ will take the form of (18), repeated here

$$\tilde{k}(t)=\tilde{k}_0+\frac{1}{2}\ \text{act}\ \cos\ \phi+\text{constant terms}\ +\ \text{periodic terms}.$$

From the second term one can see that *the parameters of the controller,* defined in (10), i.e., $k(t) = k^* + \tilde{k}(t)$, *will increase without bound.*

If there are any unmodeled dynamics at all, increasing the size of the nominal feedback controller parameters without bound will cause the adaptive system to become unstable. Indeed, since it is the gains of the nominal feedback loop that are unbounded, the system will become unstable for a large class of plants including all those whose relative degree is three or more, even if no unmodeled dynamics are present.

It should be noted that the arguments of this section have assumed that the reference input or the disturbance is purely sinusoidal. Since the adaptive system is nonlinear and the principle of superposition does not apply, the presence of a sinusoidal component in a signal will not necessarily give rise to the same effects. However, all that is needed for the argument given to apply is that a dc component result from a correlation between the output and the error. The dc component in the correlation will produce parameter drift. Thus, any disturbance whose spectrum shows frequency peaking will produce instability.

IV. Simulation Results

In this section the arguments for instability presented in the previous sections are shown to be valid via simulation.

The simulations were generated using a nominally first-order plant with a pair of complex but highly damped unmodeled poles, described by

$$y(t)=\frac{2}{(s+1)}\cdot\frac{229}{(s^2+30s+229)}[u(t)] \tag{33}$$

and a reference model

$$y_M(t)=\frac{3}{s+3}[r(t)]. \tag{34}$$

The simulations were all initialized with

$$k_y(0)=-0.65;\ k_r(0)=1.14 \tag{35}$$

which yield a stable linearization of the associated error equations. For the parameter values of (35), one finds that

$$\frac{g^*B^*(s)}{A^*(s)}=\frac{527}{s^3+31s^2+259s+527}. \tag{36}$$

The reference input signal was chosen based upon the discussion of Section III-B-2

$$r(t)=0.3+1.85\ \sin\ 16.1t, \tag{37}$$

the frequency 16.1 rad/s being the frequency at which the plant and the transfer function in (42), i.e., $g^*B^*(s)/k_r^*A^*(s)$, have 180° phase lag. A small dc offset was provided so that the linearized system would be asymptotically stable. The relatively large amplitude, 1.85 of the sinusoid in (37) was chosen so that the unstable behavior would occur over a reasonable simulation time. The adaptation gains were set equal to unity.

A. Sinusoidal Reference Inputs

Fig. 6 shows the plant output and parameters $k_r(t)$ and $k_y(t)$ for the conditions described so far. The amplitude of the plant output at the

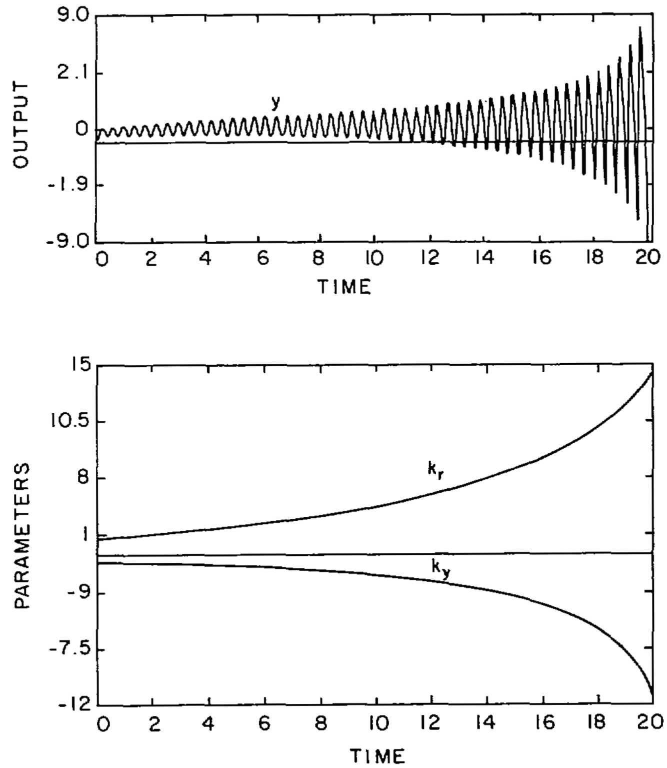

Fig. 6. Simulation of CA1 with unmodeled dynamics and $r(t) = 0.3 + 1.85 \sin 16.1 t$. (System becomes unstable.)

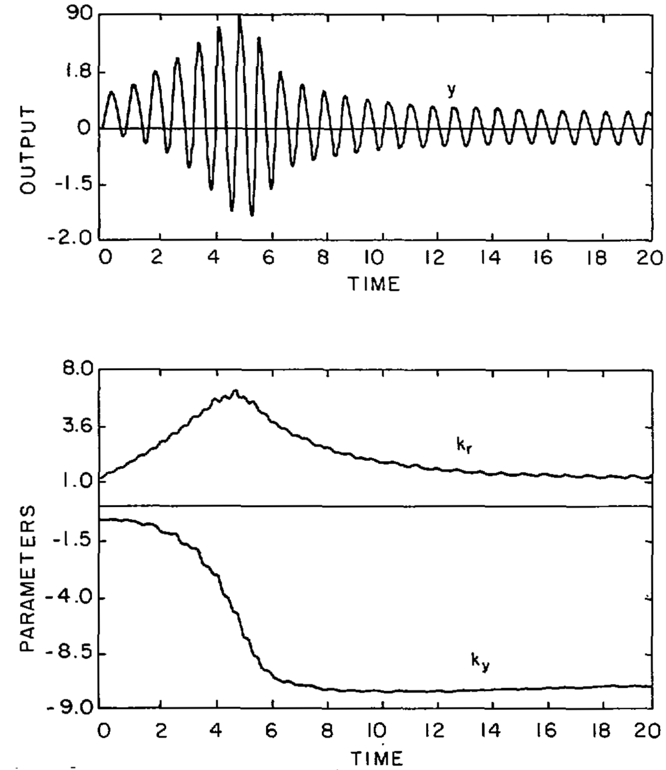

Fig. 7. Simulation of CA1 with unmodeled dynamics and $r(t) = 0.3 + 2.0 \sin 8.0 t$. (No instability observed.)

critical frequency ($\omega_0 = 16.1$ rad/s) and the parameters grow linearly with time until the loop gain of the error system becomes larger than unity. At this point in time, even though the parameter values are well within the region of stability for the linearized system, highly unstable behavior results.

Fig. 7 shows the results of a simulation, this time with the reference input

$$r(t) = 0.3 + 2.0 \sin 8.0t. \tag{38}$$

This simulation demonstrates that, if the sinusoidal input is at a frequency for which the nominal controlled plant does *not* generate a large phase shift, the algorithm may stabilize despite the high gain operator.

Similar results were obtained for the algorithms described in [3], [4], [6], [7], [9], but are not included here due to space considerations. The reader is referred to [15] for a more comprehensive set of simulation results, in which instability occurs for various sinusoidal inputs.

B. Simulations with Output Disturbances

The results in this subsection demonstrate that the instability mechanism explained in Section III-B-1 does indeed occur when there is an additive unknown output disturbance at the wrong frequency, entering the system as shown in Fig. 1. In addition, the instability mechanism of Section III-B-2 which will drive the algorithms unstable when there is a sinusoidal disturbance *at any frequency*, is also shown to take place. The same numerical example is employed here as well.

Instability Via the Phase Mechanism of Section III-B-1: In this case, CA1 was driven by a constant reference input

$$r(t) = 2.0 \tag{39}$$

with an output additive disturbance

$$d(t) = 0.5 \sin 16.1t. \tag{40}$$

The results are shown in Fig. 8, and instability occurs as predicted.

Instability Via the Gain Increase Mechanism of Section III-B-3: Fig. 9 shows the results of a simulation of CA1 that was generated with

$$r = 2.0$$

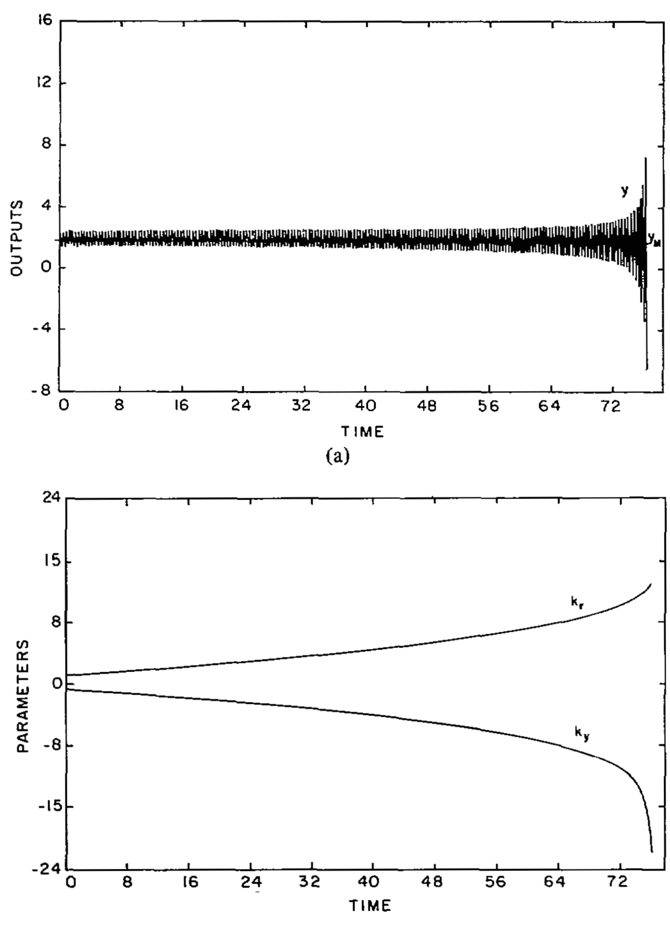

Fig. 8. Simulation of CA1 with highly damped unmodeled dynamics, $r(t) = 2.0$ and $d(t) = 0.5 \sin 16.1 t$. (System becomes unstable.)

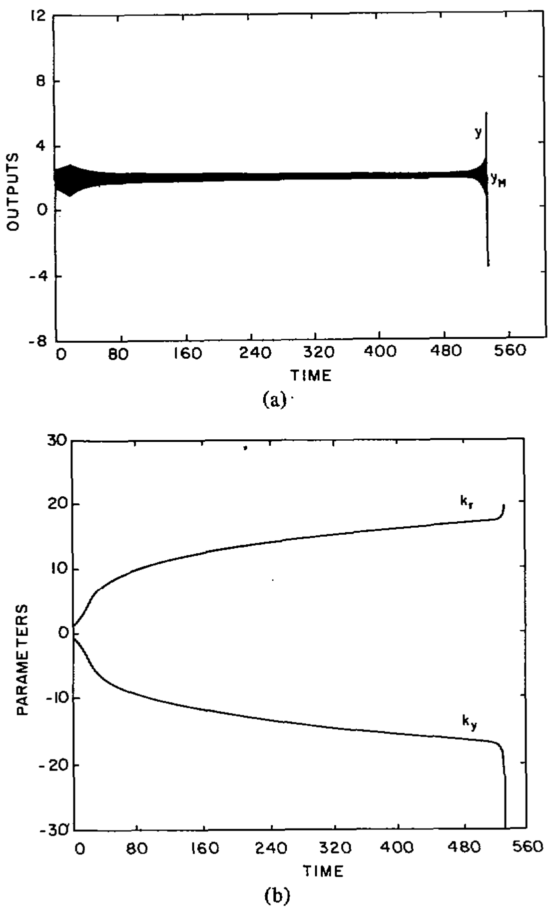

Fig. 9. Simulation of CA1 with highly damped unmodeled dynamics, $r(t)$ = 2.0 and $d(t)$ = 0.5 sin 8 t. (System becomes unstable.)

but the disturbance was changed to

$$d(t) = 0.5 \sin 8t. \quad (41)$$

The sinusoidal error signal of increasing amplitude, which is characteristic of instability via the mechanism of Section III-B-1, is not seen in Fig. 9. What is seen is that the system becomes unstable by the mechanism of Section III-B-2. While the output appears to settle down to a steady-state sinusoidal error, the k_y parameter drifts away until the point where the controller becomes unstable. (Only the onset of unstable behavior is shown in Fig. 9 in order to maintain scale.)

The most disconcerting part of this analysis is that none of the systems analyzed with constant set point reference inputs have been able to counter this parameter drift for a sinusoidal disturbance at any frequency tried!

Indeed, Fig. 10 shows the results of a simulation run with reference input

$$r = 0.0 \quad (42)$$

and *constant* disturbance

$$d = 3.0. \quad (43)$$

The simulation results show that the output settles for a long time with nonzero error but the parameter k_y increases in magnitude until instability ensues. Thus, the adaptive algorithm shows no ability to act even as a regulator when there are output disturbances.

However, in order to drive the system unstable with a constant disturbance in the same order of magnitude of time as it took to drive the system unstable with a higher frequency sinusoid, the magnitude of the disturbance must be much larger in the constant disturbance case. The constant disturbance must be larger because the nominal control system has a larger gain at dc and thus better disturbance rejection. The time it takes for the system to go unstable is inversely proportional to the square of the magnitude of the part of disturbance still present at the output of the closed-loop controlled system.

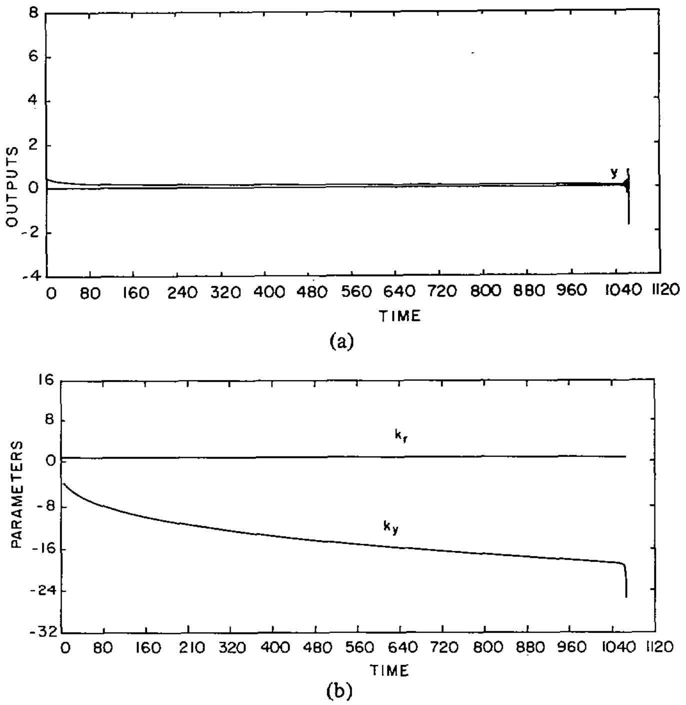

Fig. 10. Simulation of CA1 with highly damped unmodeled dynamics, $r(t)$ = 0.0 and $d(t)$ = 3.0. (System becomes unstable.)

In the previous simulations the unmodeled dynamics which were used were highly damped. Fig. 11 shows the results of a simulation of a plant with less well damped unmodeled dynamics at a somewhat lower frequency and a smaller disturbance. The plant used in the run is described by

$$y(t) = \frac{2}{s+1} \cdot \frac{100}{s^2+8s+100} [u(t)]. \quad (44)$$

The reference input was again

$$r(t) = 2.0. \quad (45)$$

The disturbance used was

$$d(t) = 0.1 \sin 8t. \quad (46)$$

In this case the parameter drift is slower than in Fig. 9. However, the parameters need not drift as far to cause instability due to the less benign unmodeled dynamics in this case, so the system exhibits unstable behavior in approximately the same amount of time.

It is important to understand that the parameters will drift and cause instability in these cases no matter how small the disturbances. If the disturbance is made small by filtering, the parameters will drift slowly and the system will take a long time to become unstable but *it will indeed become unstable* if the disturbance persists long enough. Also a sinusoidal disturbance *at any frequency* will cause parameter drift.

It appears that the *only* way to counteract the drift cause by sinusoidal disturbance is to swamp out the effects of the disturbance with probing reference inputs. These inputs must be more than sufficiently exciting. Further research is needed to quantify the effects of probing signals and develop practical guidelines as to their use.

V. Conclusions

In this paper it was shown, by analytical methods and verified by simulation results, that existing adaptive algorithms as described in [1]–[4], [6], [7], [19], have imbedded in their adaptation mechanisms infinite

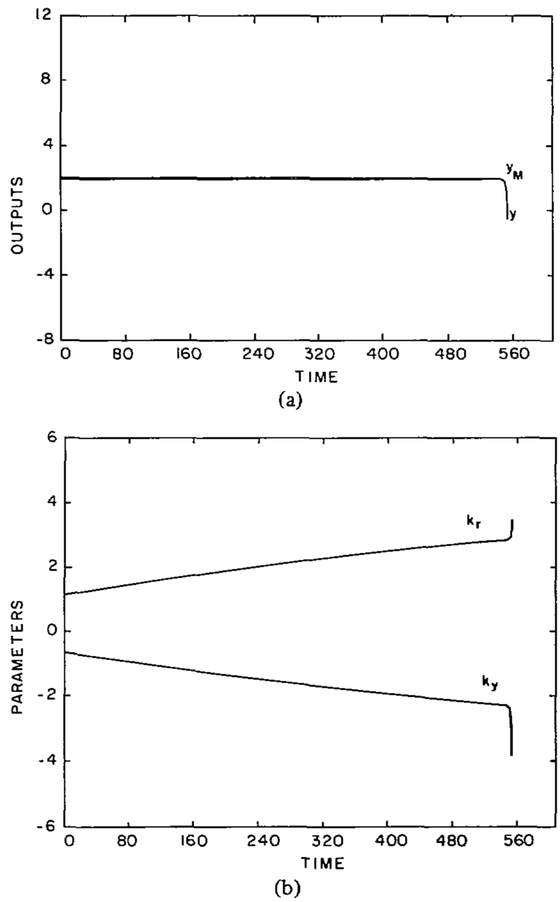

Fig. 11. Simulation of CA1 with poorly damped unmodeled dynamics, $r(t) = 2.0$ and $d(t) = 0.1 \sin 8\,t$. (System becomes unstable.)

gain operators which, in the presence of unmodeled dynamics, can cause the following:

- instability, if the reference input is a high frequency sinusoid
- instability, if there is a sinusoidal output disturbance at *any* frequency including dc.

While the first problem can be alleviated by proper limitations on the class of permissible reference inputs, the designer has no control over the additive output disturbances which impact his system. Sinusoidal disturbances are extremely common in practice and can produce disastrous instabilities in the adaptive algorithms considered.

Suggested remedies in the literature such as low-pass filtering of plant output or error signal [26], [7], [21] will not work either. It is shown in [15] that adding the filter to the output of the plant does nothing to change the basic stability problem as discussed in Section II-B. It is also shown in [15] that filtering of the output error merely results in the destabilizing input being at a lower frequency.

Exactly analogous results were also obtained for discrete-time algorithms as described in [5], [17], [18], [20] and have been reported in [15].

Finally, unless something is done to eliminate the adverse reaction to disturbances at any frequency in the presence of unmodeled dynamics, the existing adaptive algorithms should only be considered as alternatives to other methods of control in situations where stable control can be guaranteed by possibly switching to known stable backup controllers or by implementing other "safety nets." A deeper understanding of the problems presented here may lead to increased understanding of what "safety nets" are applicable and reasonable.

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A Commentary on the C. E. Rohrs *et al.* Paper "Robustness of Continuous-Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics"

KARL JOHAN ÅSTRÖM

Earlier versions of the above paper have been presented at several conferences since 1980. These presentations have certainly contributed towards making the sessions on adaptive control at the CDC Conferences lively and fun. The discussions have also inspired a lot of work on robustness of adaptive systems which have significantly contributed to our understanding of such systems. It is therefore with great pleasure that I see that the above paper is finally published so that the discussions can reach a wider audience.

It was demonstrated by Rohrs that a simple MRAS with two parameters will behave peculiarly when applied to a system with unmodeled high-frequency dynamics. If the reference signal is a step, then the system can be driven unstable by adding a small sinusoid with proper frequency to the reference or by adding measurement noise of any frequency. The paper explains the observed phenomena by demonstrating that there are operators with infinite gain in the adaptive loop. It is then pragmatically implied that this will generically lead to difficulties.

It is my opinion that this argument does not capture all the aspects of the problem. Given this opportunity I would therefore offer another explanation of the observed phenomena. The key elements of the argument are that parameters cannot be determined reliably unless the input signal is persistently exciting of the appropriate order [1]. In the presence of unmodeled dynamics it is also critical that excitation is achieved by signals in the proper frequency range. From this viewpoint the source of the difficulties is that the reference signal is a step which is persistently exciting of order one. This means that only one parameter can be determined consistently. When two parameters are adjusted they may end up anywhere on a curve in parameter space and they may drift along this curve due to disturbances. In the presence of unmodeled dynamics the parameters may then become so large that the system becomes unstable. The difficulties can be overcome if the input is a sinusoid which is persistently exciting of order two. Two parameters can then be determined. Unmodeled dynamics will not cause any difficulties if the frequency of the reference signal is sufficiently low and sufficiently large in magnitude to overcome destabilizing effects from high-frequency inputs or noise.

The heuristic arguments given above are supported by analysis and simulations in [2] and [3]. Differential equations which approximately describe the dynamics of the adaptation gains are derived using averaging methods. It is shown that these equations have an equilibrium line when the reference is a step and that the parameters will drift along this line when there are disturbances. It is also shown that the equilibrium is a point for sinusoidal command signals. The equilibrium and its local stability depend on the frequency of the reference signal.

It may justifiably be questioned if the properties of the reference signal can be postulated. If this cannot be done the excitation of the input signal can be monitored. If the signal is not properly exciting, perturbations can be added, as suggested by dual control theory, or the adaptation be switched off [4]. Variations of such schemes are found in several practical adaptive systems [5]. They may be viewed as implementations of the common sense rule "don't fit a model to bad data."

In closing I would like to thank you, Charles, for sticking your neck out as a young Ph.D. and challenging "the adaptive establishment." I have personally enjoyed our discussions. I have learned a lot from them and from trying to understand what happens in your simulations.

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On the Stability of Parallel MRAS with an AEF

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Abstract—**The stability of the parallel model reference adaptive system (MRAS) with an adjustable error filter (AEF) is discussed. It is shown that the null solutions of the parameter error vector and output error are asymptotically stable if the parameter update recursion employs time-varying adaptation gain matrix. The analysis proceeds with simple algebra, eliminates the strict positive realness (SPR) machinery, and does not resort to Lyapunov's direct method and hyperstability.**

I. Introduction

In parallel MRAS, the adjustable system is described by the difference equation

$$y_k=\sum_{i=1}^{n}\hat{a}_i(k)y_{k-i}+\sum_{i=0}^{m}\hat{b}_i(k)u_{k-i}\triangleq\underline{\hat{p}}_k^T\underline{s}_{k-1} \tag{1.1}$$

where

$$\underline{\hat{p}}_k^T=[\hat{a}_1(k),\ \cdots,\ \hat{a}_n(k),\ \hat{b}_0(k),\ \cdots,\ \hat{b}_m(k)]$$

$$\underline{s}_{k-1}^T=[y_{k-1},\ \cdots,\ y_{k-n},\ u_k,\ u_{k-m}].$$

$\{u_i\}$ and $\{y_i\}$ are the input and output sequences.

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