Towards a Unified Theory of Parameter Adaptive Control: Tunability

A. STEPHEN MORSE, FELLOW, IEEE

Abstract—**A conceptual framework is proposed in which a parameter adaptive control system is taken to be a feedback interconnection of a process Σ_P and a parameterized controller $\Sigma_C(k)$ whose parameter vector k is adjusted by a tuner Σ_T. The framework is general enough to encompass almost all of the existing continuous-time parameter adaptive control algorithms proposed in the literature for stabilizing linear process models. Emphasis is placed on the importance to adaptation of one of $\Sigma_C(k)$'s outputs called a *tuning error* e_T, which is the main signal driving Σ_T. For the closed-loop parameterized system $\Sigma(k)$ consisting of Σ_P and $\Sigma_C(k)$, definitions and characterizations are given of the concepts of *weak tunability* and *tunability* of $\Sigma(k)$ on a subset $\mathcal{E}$ of the parameter space $\mathcal{P}$ in which k takes values. It is proved that it is necessary to know a subset $\mathcal{E}$ on which $\Sigma(k)$ is weakly tunable in order to be able to construct a tuner Σ_T which adaptively stabilizes $\Sigma(k)$. Using a simple argument based on the concept of an output injection, it is proved that if $\Sigma(k)$ is tunable on $\mathcal{E}$ and Σ_T is a "gradient-like" tuning algorithm with certain typical properties, then the closed-loop adaptive control system consisting of $\Sigma(k)$ and Σ_T is "internally stable."**

I. Introduction

A CENTRAL goal of research in adaptive control in recent years has been to develop algorithms for which closed-loop stability can be established under process model assumptions which are as weak as possible. For single-input single-output continuous time-processes, these assumptions have typically been made in terms of the process model transfer function, written in the form $g\alpha(s)/\beta(s)$, where $\alpha(s)$ and $\beta(s)$ are monic coprime polynomials and g is a nonzero constant called the high frequency gain. The *classical process model assumptions* are that: 1) $\alpha(s)$ is stable (i.e., the process is minimum phase); and that 2) the sign of g, 3) the relative $n^* = \text{degree}(\beta(s)) - \text{degree}(\alpha(s))$, and 4) a bound $n \geq \text{degree}(\beta(s))$ are all known. The first adaptive control algorithm shown to be capable of globally stabilizing any process satisfying these assumptions was developed in 1978 [1]. In the intervening years, a number of significantly improved algorithms (e.g., [2]–[5]) have been found for stabilizing processes satisfying these assumptions. In addition, recent work [6]–[15] has shown that these assumptions can be relaxed very much further while still achieving stability. Still missing, however, is a conceptual framework for systematically describing these results and others in a unified way.

The main objective of this paper is to propose such a framework. The basic idea (Section II) is both natural and transparent: a parameter adaptive control system is simply the interconnection of a process, a "parameterized controller," and a "tuner." The parameterized controller controls the process and the tuner tunes the parameterized controller. "Forced" tuning takes place only when a suitably defined "tuning error," generated by the parameterized controller, is nonzero. Within this framework, the concept of *weak tunability* (Section III) can be defined, characterized, and shown to be necessary for adaptive stabilization. Our main result (Section IV) is to prove that a slightly stronger property called tunability is sufficient for adaptive stabilization, provided a tuning algorithm is used which possesses certain typical capabilities.

A. Preliminaries

Throughout this paper, $\mathbb{R}^{n\times q}$ is the real linear space of $n \times q$ matrices. For $M = [m_{ij}] \in \mathbb{R}^{n\times q}$, M' denotes its transpose and $|M| = \sum_{i=1}^{n}\sum_{j=1}^{q}|m_{ij}|$; on occasion, we also use the Euclidean norm $\|M\| = (\sum_{i=1}^{n}\sum_{j=1}^{q}|m_{ij}|^2)^{1/2}$. In general, $\|M\| \leq |M|$ and $|M| \leq \sqrt{nq}\|M\|$.

We shall make use of the following descriptive terminology, applicable to functions defined on a (possibly infinite) time interval $[0, \bar{t})$. A piecewise-continuous function $g:[0, \bar{t}) \rightarrow \mathbb{R}^n$ is in $\mathcal{L}^i$, i being a positive integer, if

$$\int_0^{\bar{t}} |g(\mu)|^i \, d\mu < C \tag{1}$$

for some constant C. If for each number $\lambda > 0$ there is a constant C, such that

$$\int_0^t e^{-\lambda(t-\mu)}|g(\mu)| \, d\mu \leq C \qquad t \in [0, \bar{t}) \tag{2}$$

then g is *bounding*. A bounding function is a *zeroing function* if $\bar{t} = \infty$ and for all $\lambda \in (0, \infty)$, $\lim_{t\rightarrow\infty} \int_0^t e^{-\lambda(t-\mu)}|g(\mu)| \, d\mu = 0$. We remark that bounding functions include functions in $\mathcal{L}^1$, $\mathcal{L}^2$, and $\mathcal{L}^\infty$, whereas functions in $\mathcal{L}^1$ or $\mathcal{L}^2$, defined on $[0, \infty)$, are zeroing functions as are piecewise-continuous functions which themselves tend to zero as $t \rightarrow \infty$ [16].

Let $x:[0, \bar{t}) \rightarrow \mathbb{R}^n$ be continuous; a bounding function g is said to be *nondestabilizing along x with growth rate* $\lambda^* \geq 0$ if: 1) for each $\lambda > \lambda^*$ there exists finite numbers C_1 and C_2 such that

$$\int_\tau^t |g(\mu)| \, d\mu \leq \lambda(t-\tau) + C_1 + \int_\tau^t \frac{C_2}{1+|x(\mu)|} \, d\mu, \qquad 0 \leq \tau \leq t \leq \bar{t} \tag{3}$$

and 2) λ^* is the least nonnegative number with this property.[1] If, in addition, for each $\lambda > \lambda^*$, (3) holds with $C_2 = 0$, then g is *nondestabilizing with growth rate* λ^*. Note that $\mathcal{L}^1$ functions are nondestabilizing with growth rate 0 since any function satisfying (1) with $i = 0$ satisfies (3) for $\lambda > 0$ with $C_1 = C$ and $C_2 = 0$. $\mathcal{L}^2$ functions are also nondestabilizing with growth rate 0 since for any $\lambda > 0$, $|g| \leq g^2/4\lambda + \lambda$ so if g satisfies (1) with $i = 2$, then (3) holds with $C_1 = C/4\lambda$ and $C_2 = 0$.

A piecewise-continuous matrix $A:[0, \bar{t}) \rightarrow \mathbb{R}^{n\times n}$ is *exponentially stable* if there exist positive numbers C and λ

[1] The inclusion in (3) of the integral involving $|x(\cdot)|$ is prompted by the results of [5].

Manuscript received October 5, 1988; revised November 7, 1988, March 31, 1989, January 10, 1990, and April 24, 1990. Paper recommended by Associate Editor, P. A. Ioannou. This work was supported by the United States Air Force Office of Scientific Research under Grant 84-0242 and by the National Science Foundation under Grant ECS-8805611.

The author is with the Department of Electrical Engineering, Yale University, New Haven, CT 06520-2157.

IEEE Log Number 9036929.

0018-9286/90/0900-1002$01.00

such that the state transition matrix $\phi_A(t,\tau)$ of $A(t)$ satisfies $|\phi_A(t,\tau)| \le Ce^{-\lambda(t-\tau)}$, $t, \tau \in [0, \bar{t})$. Note, that if A is exponentially stable and $g:[0,\bar{t}) \to \mathbb{R}^n$ is nondestabilizing along x with growth rate λ^*, then $\int_0^t \phi_A(t,\tau)g(\tau)\,d\tau$ is nondestabilizing along x with growth rate $C\lambda^*$ since this integral is bounded by $C\int_0^t e^{-\lambda(t-\tau)}|g(\tau)|\,d\tau$, which, in turn, is bounded by $C\int_0^t |g(\tau)|\,d\tau$.

A function $f:\Omega \to \mathbb{R}^{n\times m}$ on an open set $\Omega \subset \mathbb{R}^{p\times q}$ is continuously differentiable if it possesses a continuous first derivative; f is locally Lipschitz if for each compact (i.e., closed, bounded) subset $\bar{\Omega} \subset \Omega$, there exists a constant L depending only on $\bar{\Omega}$ for which $|f(x) - f(y)| \le L|x-y|$, all $x, y \in \bar{\Omega}$. It is known [17], [18] that continuously differentiable functions are locally Lipschitz and that locally Lipschitz functions are continuous.

We shall make use of certain basic facts concerned with the nonlinear differential equation

$$\dot{x} = \alpha(x, t) \tag{4}$$

where $\alpha:\mathbb{R}^n x[0,\bar{t}) \to \mathbb{R}^n$ is locally Lipschitz in x and piecewise continuous in t. It is known [17], [18] that for each initial state $x_0 \in \mathbb{R}^n$, there is a maximal interval $[0,\hat{t}) \subset [0,\bar{t})$ on which a unique continuous solution $y(t)$ to (4) satisfying $y(0) = x_0$ exists; and if $|y(t)|$ is bounded on this interval, then $\hat{t} = \bar{t}$ and the solution necessarily is bounded on $[0,\bar{t})$. In some applications, (4) is of the special form

$$\dot{x}_1 = A_1(t)x_1 + f_1(x_1, x_2, t) + b(t)$$

$$\dot{x}_2 = A_2(t)x_2 + f_2(x_1, t) \tag{5}$$

where the A_i are exponentially stable, b is bounded, and the f_i are nonlinear functions. The following lemma asserts that if along a solution $y = [y_1', y_2']'$ to (5), the f_i satisfy

$$\left.\begin{aligned} &|f_1(y_1(t), y_2(t), t)| \le \sigma(t)(|y_1(t)| + C_1|y_2(t)|) + h(t) \\ &|f_2(y_1(t), t)| \le C_1|y_1(t)| + C_2 \end{aligned}\right\} \quad t \in [0,\hat{t}) \tag{6}$$

where h is bounding, the C_i are nonnegative constants, and σ is nondestabilizing with sufficiently small growth rate, then y itself must exhibit bounding limiting behavior.

Lemma 1: Let C and λ be positive constants for which the state transition matrix ϕ_{A_i} of A_i satisfies $|\phi_{A_i}(t,\tau)| \le Ce^{-\lambda(t-\tau)}$, $0 \le \tau \le t \le \bar{t}$, $i = 1, 2$. If h is bounding and σ is nondestabilizing along y with growth rate λ^* and $\lambda^* + 2C_1\sqrt{\lambda^*} < \lambda/C$, then $y(t)$ exists and is bounded on $[0,\bar{t})$. If, in addition, $\bar{t} = \infty$ and σ and h are zeroing functions, then as $t \to \infty$, $y_1(t)$ approaches the unique bounded solution $x^*(t)$ to the equation

$$\dot{x}^* = A_1(t)x^* + b(t) \tag{7}$$

$$x^*(0) = 0. \tag{8}$$

Although the proof of this lemma can easily be deduced from existing literature (e.g., [19], [4], [22]), for completeness, a proof is given in the Appendix.

II. Conceptual Framework

Classically, [1]–[3] parameter adaptive control systems have been defined and discussed in terms of error models in which parameters typically enter linearly. It turns out that for many adaptive algorithms (e.g., [6], [8], [9], [12]–[15], [29]) error models play no essential role and parameters enter nonlinearly. Hence, a new conceptual framework is needed to describe both the classical and more recent adaptive structures. To construct such a framework it is useful to think of a *parameter adaptive control system* as a system consisting of three components: a process, a parameterized controller, and a tuner, Fig. 1. The *process* is a dynamical system with control input $u \in \mathbb{R}^{n_u}$, disturbance input $w \in \mathbb{R}^{n_w}$, and measured output $y \in \mathbb{R}^{n_y}$. The *parameterized controller* is a dynamical system $\Sigma_C(k)$ depending on a control parameter k which takes values in some subset $\mathcal{P} \subset \mathbb{R}^{n_p}$. The inputs to $\Sigma_C(k)$ are the tuner output k, the process output y, and possibly a reference input r. The outputs generated by $\Sigma_C(k)$ are the closed-loop control input u to the process, a *tuning error* e_T, as well as supplementary tuning data d consisting of known functions of r, y, and the parameterized controller's state. The tuner is an algorithm $\Sigma_T(p_I)$, *initialized* by p_I (i.e., $k(0) = p_I$), with inputs d and e_T and output k, $k(t)$ being the "tuned value" of k at time t; $\Sigma_T(p_I)$ is presumed to be well defined for each $p_I \in \mathcal{P}$.

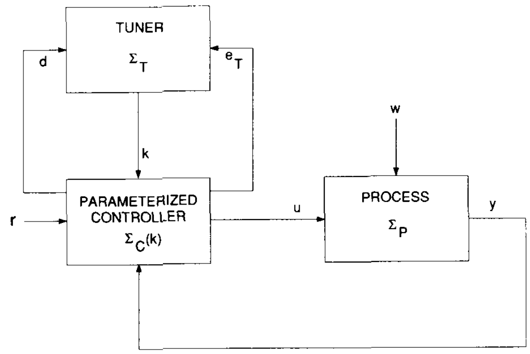


Fig. 1. A parameter adaptive control system.

The function of Σ_T is to adjust k to make e_T "small" in some suitably defined sense. Although the specifics of Σ_T may vary greatly from algorithm to algorithm, in most instances, tuning is carried out in one of two fundamentally different ways, depending on whether $\mathcal{P}$ is countable or not. For the countable case (e.g., see [7], [12], [29]), tuning is achieved by sequentially stepping k through $\mathcal{P}$ along a predetermined path, using on-line (i.e., real-time) data to decide only when to switch k from one value along the path to the next. In contrast, for the uncountable case (e.g., see [1]–[5], [10], [11], [20]–[23]) the path in $\mathcal{P}$ along which k is adjusted is not determined off-line but instead is computed in real time from "gradient-like" data. The main advantage of countable search algorithms over gradient-like procedures appears to be their broader applicability. On the other hand, when applicable, gradient-like algorithms are likely to exhibit far superior performance, but so far, this has not been clearly demonstrated.

Since a tuner Σ_T is an algorithm driven by e_T, Σ_T will typically possess certain "rest" or "equilibrium" values of k at which tuning ceases if $e_T = 0$. To be more precise, let us agree to say that (in open loop) a tuner $\Sigma_T(p_I)$ is at *equilibrium value* $p_0 \in \mathcal{P}$ at time $t_0 \ge 0$, if $k(t_0) = p_0$ and if for the input $e_T(t) = 0$, $t \ge t_0$, $k(t)$ remains fixed at p_0 for $t \ge t_0$. In this paper, we assume that Σ_T is stationary to the extent that its possible equilibrium values at t_0 are independent of t_0 and $p_I \in \mathcal{P}$; and we define the tuner's *equilibrium set* $\mathcal{E}_T$ to be the set of all such values in $\mathcal{P}$.

One fairly general algorithm for a tuner might be a dynamical system of the form

$$\dot{k} = \pi_1(k) + \pi_2(k, d, x_N)e_T$$

$$\dot{x}_N = \pi_3(k, d, x_N) \tag{9}$$

$$k(0) = p_I$$

$$x_N(0) = x_{N_0} \tag{10}$$

where the $\pi_i(\cdot)$ are nonlinear functions, x_N is the state of a "dynamic normalizer" (cf. Section IV), $p_I \in \mathbb{R}^{n_p}$ and $x_{N_0} \in \mathbb{R}^{n_N}$.

In this case, $\mathcal{P}$ can be taken as $\mathbb{R}^{n_p}$ and $\mathcal{E}_T = \{p: \pi_1(p) = 0\}$. Alternatively, Σ_T might be a switching algorithm of the form

$$\left.\begin{array}{ll} k(t) = p_I, & t \in [t_0, t_1) \\ k(t) = h(i), & t \in [t_i, t_{i+1}), \quad i \geq 1 \end{array}\right\} \quad (11)$$

where h is a function from the positive integers to $\mathbb{R}^{n_p}$, $p_I \in$ image h, $t_0 = 0$, and for $i > 0$

$$t_i = \begin{cases} \min\limits_{t > t_{i-1}} \left\{ t: \int_0^t \|e_T\|^2 \, d\tau = (i!) \right\}, & \text{if this set is nonempty} \\ & \text{or} \\ \infty, & \text{otherwise} \end{cases}$$

[29]; in this case, $\mathcal{P}$ is the image of h (which is countable), and $\mathcal{E}_T = \mathcal{P}$. The ideas which follow apply to both types of algorithms. Thus, until Section IV, no special assumptions will be made about Σ_T, other than that it be stationary and possess a nonempty equilibrium set.

In the sequel, we assume that the process can be modeled by a stationary finite-dimensional linear system Σ_P with state x_P. We take Σ_P to be a member of some family of systems $\mathfrak{M}$ continuously parameterized by an *uncertainty vector* q which takes values in some open subset $\mathbb{Q} \subset \mathbb{R}^{n_q}$. The models in $\mathfrak{M}$ are of the form

$$\begin{aligned} \dot{x}_P &= A_P(q)x_P + B_P(q)u + \bar{B}_P(q)w \\ y &= C_P(q)x_P + D_P(q)w \end{aligned} \quad (12)$$

where $A_P, B_P, \cdots, D_P$ are continuous matrix valued functions on $\mathbb{Q}$. For simplicity, we choose our parameterization so that $0 \in \mathbb{Q}$ and view (12) evaluated at $q = 0$ as a *nominal process model* Σ_N; sometimes Σ_N's transfer matrix from u to y is known exactly, but we shall not make use of this fact here. The vector $q_P \in \mathbb{Q}$, for which (12) models Σ_P is sometimes called the "mismatch error" (cf. [4]) and as such is a characterization of Σ_P's "structured uncertainty." More generally, q_P might consist of two subvectors, one a mismatch error, the other a vector of numbers $\mu_1, \mu_2 \cdots$ weighting Σ_P's "unmodeled dynamics" (cf. [4]). In what follows it is not necessary to be too specific about the definition of q or about the way $A_P, \cdots, D_P$ depend on q, except for the assumption, which has already been made, that this dependence is continuous.

The parameterized controller $\Sigma_C(k)$ will be taken to be a stationary dynamical system of the form

$$\begin{aligned} \dot{x}_C &= A_C(k)x_C + B_y(k)y + B_r(k)r \\ u &= F_C(k)x_C + F_y(k)y + G_r(k)r \\ e_T &+ C_C(k)x_C + C_y(k)y + D_r(k)r \\ d &= E_C(k)x_C + E_y(k)y + E_r(k)r \end{aligned} \quad (13)$$

where $A_C(\cdot), \cdots, E_r(\cdot)$ are matrix-valued functions on some parameter space $\mathcal{P} \subset \mathbb{R}^{n_p}$.

The preceding framework is general enough to encompass almost all of the existing continuous-time parameter adaptive control algorithms proposed in the literature for stationary linear process models. For example, the classical adaptive control systems studied in [2], [3] admit this characterization; in these cases, e_T is the (unnormalized) augmented error originally introduced by Monopoli [30] and $\mathcal{E}_T = \mathbb{R}^{n_p}$. Systems using "high-gain" adaptive stabilizers such as those in [7], [9], [13]–[15] can also be described in this way; in these cases, k is a scalar, $\mathcal{E}_T = \mathbb{R}$, and e_T might be y, $[y', u']'$, or $[y', x_C']'$. Similarly, the switching algorithms of [7], [12], [29] are covered by this framework, as has already been noted. It is also possible to use this framework to describe a large variety of robustly stabilized adaptive systems such as those studied in [4], [5]; for example, for a system utilizing the algorithm of [4], e_T would be an (unnormalized) augmented error and $\mathcal{E}_T$ would be a compact subset of $\mathcal{P}$. Most "indirect" adaptive control algorithms (see [22]) also admit this characterization—in these cases, e_T is typically an unnormalized identification error, and $\mathcal{E}_T = \mathcal{P} = \mathbb{R}^{n_p}$. In fact, just about the only continuous-time algorithm we know of which does not quite fit this framework, is the one studied in [1]. This is because the parameterized controller $\Sigma_C(k)$ of [1] includes in its defining equation for u a term of the form $N(k, d)e_T$, N being a nonlinear function of k and d. Notwithstanding this difference the concepts and results of Section III apply to this adaptive system as well.

III. Tunability

The *closed-loop parameterized system* $\Sigma(p)$ determined by (12) and (13) can be concisely described by a system of equations of the form

$$\begin{aligned} \dot{x} &= A(p)x + B(p)v \\ e_T &= C(p)x + D(p)v \\ d &= E(p)x + G(p)v \end{aligned} \quad (14)$$

where $x = [x_P', x_C']'$ and $v = [r', w']'$. Here $A, \cdots, G$ are defined in the obvious way using (12) and (13).[2] In this setting, the following question arises. What must be true of $\Sigma(p)$ in order for there to exist a tuning algorithm Σ_T [e.g., (9), (10)] for which the closed-loop adaptive system consisting of Σ_T and $\Sigma(k)$ is "stable"? In the sequel, we provide some preliminary answers to this question.

With $\mathcal{E}$ any fixed, nonempty subset of $\Sigma(p)$'s parameter space $\mathcal{P}$, let us agree to call (14) *weakly tunable on* $\mathcal{E}$, if for each fixed $p \in \mathcal{E}$ and each bounded, piecewise-continuous exogenous input $v: [0, \infty) \to \mathbb{R}^{n_v}$, every possible system trajectory for which $e_T(t) = 0$, $t \in [0, \infty)$, is bounded on $[0, \infty)$. Call $\Sigma(p)$ *tunable on* $\mathcal{E}$ if for each $p \in \mathcal{E}$, x goes to zero as $t \to \infty$ whenever both e_T and v equal zero on $[0, \infty)$.

Remark 1: It is easy to verify that $\Sigma(p)$ is weakly tunable on $\mathcal{E}$, just in case, for each $p \in \mathcal{E}$, the matrix pair $(C(p), A(p))$ is weakly detectable[3] and the matrix pair obtained by restricting $C(p)$ and $A(p)$ to the controllable space of $(A(p), B(p))$ is detectable. Similarly, $(C(p), A(p))$ is tunable on $\mathcal{E}$ if and only if, $C((p), A(p))$ is detectable for each $p \in \mathcal{E}$. Thus, tunability of $\Sigma(p)$ on $\mathcal{E}$ implies weak tunability of $\Sigma(p)$ on $\mathcal{E}$ and is equivalent to weak tunability of $\Sigma(p)$ on $\mathcal{E}$ whenever $\Sigma(p)$ is controllable on $\mathcal{E}$.

Our aim here is to briefly explain why weak tunability is necessary for adaptive stabilization. To be specific, call $\Sigma_T(\cdot)$ an *unbiased stabilizer* of $\Sigma(k)$ if for each initialization $p_I \in \mathcal{P}$, each bounded piecewise-continuous exogenous input $v: [0, \infty) \to \mathbb{R}^{n_v}$ and each initial state $x(0)$, the state response x of $\Sigma(k)$, tuned by $\Sigma_T(p_I)$, is bounded on $[0, \infty)$.

Suppose Σ_T is a candidate tuner for $\Sigma(p)$. The definition of weak tunability implies that if $\Sigma(p)$ is not weakly tunable on Σ_T's equilibrium set $\mathcal{E}_T$, then for some exogenous input v, initial state $x(0)$, and parameter value $p_0 \in \mathcal{E}_T$, the untuned system $\Sigma(p_0)$ will admit an unbounded state response $x(t)$ along which $e_T(t) = 0$. If the same input v is applied to $\Sigma(k)$, with k tuned by $\Sigma_T(p_0)$, then clearly, $k(t) = p_0$, $t \geq 0$ and $e_T(t) = 0$, $t \geq 0$.

[2] For clarity, we have not explicitly denoted the continuous dependence of these matrices on the process model uncertainty vector q.

[3] A matrix pair (C, A) is *weakly detectable* if for each vector x for which $Ce^{At}x$ is bounded on $[0, \infty)$, it follows that $e^{At}x$ is bounded on $[0, \infty)$, as well. (C, A) is *detectable* if for each eigenvalue–eigenvector pair (λ, x), λ has a negative real part whenever $Cx = 0$. Detectability implies weak detectability but the converse is not necessarily true.

Therefore, $\Sigma(k)$ will have exactly the same unbounded response to v that $\Sigma(p_0)$ has. We are led to the following theorem.

Theorem 1: A necessary condition for tuner $\Sigma_T(\cdot)$ to be an unbiased stabilizer of the tuned system $\Sigma(k)$, is that $\Sigma(p)$ be weakly tunable on the tuner's equilibrium set $\mathcal{E}_T$.

Clearly, weak tunability on $\mathcal{E}_T$ is a fundamental property that any parameter adaptive control system of the aforementioned general type must have if stability is to be assured.[4] An interesting problem then, is to determine what is required of a process model Σ_p and its parameterized controller $\Sigma_C(p)$ for the resulting closed-loop system $\Sigma(p)$ to be weakly tunable or tunable on some given subset $\mathcal{E} \subset \mathcal{P}$. This problem is discussed further in [28]. In the sequel, we give some examples of tunable and untunable systems.

Example 1: Suppose for Σ_P, we take the one-dimensional system

$$\dot{y} = ay + gu \tag{15}$$

with a and g unknown constants satisfying $a > 0$ and $g \neq 0$. To stabilize this system, consider using a control law of the form

$$u = \hat{f} y \tag{16}$$

where, if we had our preference, we would choose $\hat{f}$ so that $a + g\hat{f} = -1$ since this would stabilize (15); but since a and g are unknown, we might instead try to choose $\hat{f}$ in accordance with the certainty-equivalence principle so that

$$\hat{a} + \hat{g}\hat{f} = -1 \tag{17}$$

where $\hat{a}$ and $\hat{g}$ are estimates of a and g, respectively. However, since standard identification algorithms may cause $\hat{g}$ to pass through zero, to avoid the possibility of "division by zero" let us consider in place of (17), the "gradient" adjustment law

$$\dot{\hat{f}} = -\hat{g}(\hat{a} + \hat{g}\hat{f} + 1) \tag{18}$$

as a means of generating $\hat{f}$. Finally, to construct estimates $\hat{a}$ and $\hat{g}$, observe from (15) that

$$y = (a+1)\bar{y} + g\bar{u} + \epsilon$$

where $\epsilon = e^{-t}(y(0) - (a+1)\bar{y}(0) - g\bar{u}(0))$, and

$$\dot{\bar{y}} + \bar{y} = y$$
$$\dot{\bar{u}} + \bar{u} = u. \tag{19}$$

Thus, to generate $\hat{g}$ and $\hat{a}$, it makes sense to use an algorithm driven by the "identification error"

$$e = (\hat{a}+1)\bar{y} + \hat{g}\bar{u} - y, \tag{20}$$

since this results in the familiar error equation

$$e = \bar{y}(\hat{a} - a) + \bar{u}(\hat{g} - g) - \epsilon.$$

If a standard identification algorithm is used, identification ceases when $e = 0$, in which case, $\hat{a}$ and $\hat{g}$ become constant. Viewing this algorithm together with (18) as a tuner Σ_T with tuning input $e_T = e$, and tuned parameter $[\hat{g}, \hat{a}, \hat{f}]'$, Σ_T's equilibrium set will be

$$\mathcal{E}_T = \{[p_1, p_2, p_3]' : p_1(p_2 + p_1 p_3 + 1) = 0, \quad [p_1, p_2, p_3]' \in \mathbb{R}^3\}.$$

The parameterized controller $\Sigma_C(p)$ corresponding to (16), (19), and (20) is thus

$$\dot{\bar{y}} + \bar{y} = y$$
$$\dot{\bar{u}} + \bar{u} = p_3 y$$
$$e_T = (p_2+1)\bar{y} + p_1\bar{u} - y$$
$$u = p_3 y \tag{21}$$

and $\Sigma(p)$ is the closed-loop parameterized system, described by (15) and (21).

Observe that the point $[0, a, 0]' \in \mathcal{E}_T$. It is easy to verify that for this value of p, $\Sigma(p)$ admits the unbounded solution $y = e^{at}$, $\bar{y} = e^{at}/(1+a)$, $\bar{u} = 0$, $e_T = 0$ so $\Sigma(p)$ is not weakly tunable on $\mathcal{E}_T$.

Example 2: In Example 1, $\Sigma(p)$ is untunable on $\mathcal{E}_T$ because $\mathcal{E}_T$ contains points $[\hat{g}, \hat{a}, \hat{f}]'$ for which $\hat{a} + \hat{g}\hat{f} > 0$. It is possible to eliminate this problem and to achieve stability, if sign(g) is assumed known, by using in place of (18), the adjustment law

$$\dot{\hat{f}} = -\text{sign}(g)(\hat{a} + \hat{g}\hat{f} + 1) \tag{18$'$}$$

together with tuning equations

$$\dot{\hat{a}} = -(\hat{a} + \hat{g}\hat{f} + 1) - \bar{y}e_T$$
$$\dot{\hat{g}} = -\hat{f}(\hat{a} + \hat{g}\hat{f} + 1) - \bar{u}e_T$$

where e_T is the identification error e defined by (20).[5] In this case, Σ_T's equilibrium set is precisely those points $[\hat{g}, \hat{a}, \hat{f}]' \in \mathbb{R}^3$ for which (17) holds. It is straightforward to check that at any point $[p_1, p_2, p_3]' \in \mathcal{E}_T$,

$$y(t) = p_1(\bar{u}(0) - p_3\bar{y}(0))e^{-t}$$

along any solution $[y(t), \bar{y}(t), \bar{u}(t)]'$ to (15) and (21) for which $e_T(t) \equiv 0$. Since this and (21) imply that any such solution is bounded, $\Sigma(p)$ is now tunable on $\mathcal{E}_T$.

Example 3: Take $\mathcal{P} = \mathbb{R}$ and let $\Sigma(p)$ be any parameterized system with $C(p) = [1, 0]$ and $A(p) = \begin{bmatrix} 1 & 1 \\ 0 & p \end{bmatrix}$. Since $C(p), A(p))$ is detectable on $\mathcal{P}$, by Remark 1, $\Sigma(p)$ must be tunable on each subset $\mathcal{E} \subset \mathcal{P}$. In spite of this, observe that no matter how k is tuned, $\Sigma(k)$ can have an unbounded state response (e.g., with $v = 0$ and $x(0) = [1, 0]'$, $x(t) = e^t[1, 0]'$), so adaptive stabilization is impossible.

The preceding example shows that tunability of $\Sigma(p)$ on a known subset $\mathcal{E} \subset \mathbb{R}^{n_p}$ is not sufficient to ensure that there is a tuner Σ_T which will adaptively stabilize $\Sigma(k)$. However, using the ideas of [7], it can be shown that if $\Sigma(p)$ is tunable on $\mathcal{E}$ and in addition, $\mathcal{E}$ contains a countable dense subset $\mathcal{E}^*$ as well as a parameter value p_0 for which $\Sigma(p_0)$ is internally stable, then without knowing p_0, it is possible to construct a switching algorithm Σ_T depending on $\mathcal{E}^*$ with $\mathcal{E}_T \subset \mathcal{P} \subset \mathcal{E}^*$, which is an unbiased stabilizer of $\Sigma(k)$. Thus, to achieve adaptive stability with some tuner Σ_T, it is enough to design $\Sigma_C(p)$ so that $(C(p), A(p))$ is detectable on a known subset $\mathcal{E} \subset \mathbb{R}^{n_p}$ containing a countable dense subset $\mathcal{E}^*$ and a point p_0 which stabilizes $A(p_0)$. The following example shows that this is very easy to do without assuming very much about Σ_P.

Example 4: For fixed integer $n > 0$, define parameter vector $p = [p_1, p_2, \cdots, p_{2n+1}]'$, and parameterized polynomials $\beta(p, s) = s^n + p_n s^{n+1} + \cdots p_2 s + p_1$, $\alpha(p, s) = p_{2n+1}s^n + p_{2n}s^{n-1} + \cdots p_{n+2}s + p_{n+1}$. Choose $\gamma(s)$ to be any monic sta-

[4] For algorithms utilizing persistently exciting probing signals, weak tunability on $\mathcal{E}_T$ may well be more than is required for stability. This issue will be discussed further in another paper.

[5] Motivation for these equations, which can be found in [25], stems from the observation that "control error" $e_c = (\hat{a} + \hat{g}\hat{f} + l)$ and tuning error e_T can be written, respectively, as $e_c = (\hat{a} - a) + \hat{f}(\hat{g} - g) + g(\hat{f} - f)$ and $e_T = \bar{y}(\hat{a} - a) + \bar{u}(\hat{g} - g) - \epsilon$, where $f = -(1+a)/g$.

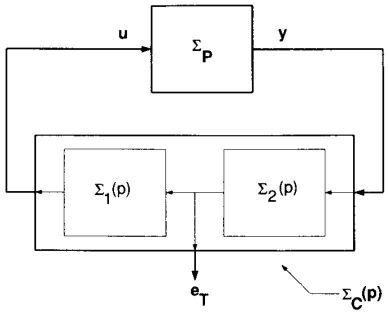


Fig. 2. A parameterized system $\Sigma(p)$.

ble polynomial of degree n and let $\Sigma_1(p) = (A_1(p), b_1(p), c_1(p), d_1(p))$ and $\Sigma_2(p) = (A_2(p), b_2(p), c_2(p), d_2(p))$ be n-dimensional realizations of $\alpha(p, s)/\gamma(p, s)$ and $\gamma(s)/\beta(p, s)$, respectively. Define $\Sigma_C(p)$ to be the cascade interconnection of Σ_2 with Σ_1 as shown in Fig. 2.

Observe that if for fixed p, e_T is identically zero, then both u and y must go to zero since $\Sigma_1(p)$ is stable and $\Sigma_2(p)$ has a stable proper inverse. From this, it follows that if Σ_P is a stabilizable and detectable process model, then for all $p \in \mathbb{R}^{2n+1}$, $\Sigma(p)$ must be detectable through e_T.[6] Let $\mathcal{E}^*$ be a countable, dense subset of $\mathbb{R}^{2n+1}$; clearly $\Sigma(p)$ is tunable on $\mathcal{E}^*$. Moreover, if in the linear space $\mathbb{R}^{1\times n_p} \oplus \mathbb{R}^{n_p \times n_p} \oplus \mathbb{R}^{n_p+1}$, Σ_p is sufficiently close to a system which is stabilizable, detectable, and of McMillan degree not exceeding n, then there must exist a vector $p_0 \in \mathcal{E}^*$ for which $A(p_0)$ is stable. Thus, $\mathcal{E}^*$ and $\Sigma(p)$ will have what is required for adaptive stabilization as long as Σ_p is stabilizable, detectable, and close enough to a stabilizable, detectable system with McMillan degree no greater than n.

IV. Tuning Theorem

In this section, we study the behavior of the parameterized system $\Sigma(k)$ described by

$$\dot{x} = A(k)x + B(k)v$$

$$e_T = C(k)x + D(k)v$$

$$d = E(k)x + G(k)v \tag{22}$$

for the case when k is continuously tuned. As before, $\Sigma(k)$ is taken to be a representation of the closed-loop interconnection of process model Σ_P and parameterized controller $\Sigma_C(k)$. We assume that $\mathcal{P}$ is an open-connected subset of $\mathbb{R}^{n_p}$, that all parameterized controller matrices in (13) are at least locally Lipschitz on $\mathcal{P}$, and that A_C, B_y, F_C, F_y, C_C, and C_y are continuously differentiable. These assumptions imply that $B:\mathcal{P} \to \mathbb{R}^{n\times n_v}$, $D:\mathcal{P} \to \mathbb{R}^{n_T \times n_v}$, $E:\mathcal{P} \to \mathbb{R}^{n_d \times n}$ and $G:\mathcal{P} \to \mathbb{R}^{n_d \times n_v}$ are locally Lipschitz and that $C:\mathcal{P} \to \mathbb{R}^{n_T \times n}$ and $A:\mathcal{P} \to \mathbb{R}^{n\times n}$ are continuously differentiable. The previously assumed continuous dependence on $q \in \mathcal{Q}$ of all process model matrices in (12) further implies that $A, \cdots, G$ depend continuously on q as do the derivatives with respect to p of $C(p)$ and $A(p)$.

As a tuner for k, we shall consider an algorithm Σ_T consisting of a *parameter adjustment law*:

$$\dot{k} = \Pi_T(k) + MW\bar{e}_T, \tag{23}$$

a *normalized tuning error*

$$\bar{e}_T = e_T - W'NW\bar{e}_T, \tag{24}$$

a *weighting matrix*

$$W = W(k, d, x_N), \tag{25}$$

[6] It is interesting to note that this will no longer be true if Σ_1 and Σ_2 are interchanged, unless Σ_P is restricted to be minimum phase.

and a *dynamic normalizer*

$$\dot{x}_N = A_N x_N + b_N(k, d). \tag{26}$$

We assume that $b_N:\mathcal{P} \times \mathbb{R}^{n_d} \to \mathbb{R}^{n_N}$ and $W:\mathcal{P} \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_N} \to \mathbb{R}^{m \times n_T}$ are locally Lipschitz, that $\Pi_T:\mathcal{P} \to \mathcal{P}$ is continuously differentiable and that M, N, and A_N are constant matrices with A_N exponentially stable and $N = N'$ positive semidefinite. We further assume that for each compact subset $\bar{\mathcal{P}} \subset \mathcal{P}$ there are constants C_{Wi} and C_{Ni} for which both W and b_N satisfy the *linear growth conditions*

$$\left.\begin{aligned} |W(p, d, x_N)| &\le C_{W1}(|d| + |x_N|) + C_{W2} \\ |b_N(p, d)| &\le C_{N1}|d| + C_{N2} \end{aligned}\right\} \cdot p \in \bar{\mathcal{P}},\ d \in \mathbb{R}^{n_d},\ x_N \in \mathbb{R}^{n_N}. \tag{27}$$

Remark: With some additional effort, it can be shown that the results which follow also apply if, instead of (27), W and b_N satisfy the growth conditions $|W| \le C_{W1}(|d| + |x_N|^{1/2}) + C_{W2}$ and $|b_N| \le C_{N1}|d|^2 + C_{N2}$, respectively.

Remark: In some tuning algorithms (e.g., see [2]), $\bar{e}_T$ is defined by equations of the form $\bar{e}_T = e_T - Cz$, $\dot{z} = Az + BW'NW\bar{e}_T$ where (C, A, B) is a square controllable observable linear system with positive real transfer matrix. It is not difficult to generalize the results which follow to encompass this alternative case.

Tuner equations as general as (23)–(26) describe a large number of algorithms including, with slight modification, all of those surveyed in [31] and [22] which continuously tune k. For a SISO process with $d = [x_C', y, r]'$, e_T an unnormalized augmented error, $W = d$, $\Pi_T = 0$, and $N = M = I_{n_d \times n_d}$, the preceding equations model one of the two algorithms examined in [2]; in this case, (26) is absent and the normalization of e_T via (24) is "nondynamic." This example can be modified to illustrate "dynamic" normalization—simply redefine N, M, and W so that $M = [I_{n_d \times n_d}, 0]$, $N = I_{n_d+1 \times n_d+1}$, and $W = [d', x_N]'$, and include the one-dimensional dynamic normalizer $\dot{x}_N = A_N x_N + |u| + |y|$. The role of dynamic normalization has been discussed in [32].

The assumption that Π_T is continuously differentiable, which is made for simplicity, is mildly restrictive, and precludes direct application of the theorem which follows to the analysis of algorithms such as the "switching σ-modified" tuner of [31], since that algorithm uses a function Π_T which is continuous and piecewise-linear, but not everywhere differentiable. In cases of this type, the difficulty can easily be avoided if one is willing to replace Π_T with a continuously differential approximation $\hat{\Pi}_T$ which is sufficiently close to Π_T to preserve its essential features. Alternatively, using ideas similar to those exploited in [20, Proof of Lemma 3], it should be possible to prove the tuning theorem which follows, assuming only that Π_T is locally Lipschitz.

Note that Example 2 utilizes a tuning algorithm which can be described by (23)–(26) with continuously differentiable Π_T. In this case, $\Pi_T([\hat{g}, \hat{a}, \hat{f}]') = e_C[-\hat{f}, -1, \operatorname{sign}(g)]'$, where $e_C = (\hat{a} + \hat{g}\hat{f} + 1)$. The tuning theorem which follows suggests that the idea of "implicit tuning" illustrated by Example 2, can be generalized considerably. This will be discussed in a future paper.

In the sequel, we study the closed-loop adaptive control system described by (22)–(26) for all values of uncertainty vector q in some compact subset $\bar{\mathcal{Q}} \subset \mathcal{Q}$ with dense interiors containing the zero vector. In view of Theorem 1 and Remark 1, we make the following assumption.

Tunability Assumption: For each $q \in \mathcal{Q}$, $\Sigma(p)$ is tunable on the equilibrium set $\mathcal{E}_T = \{p: \Pi_T(p) = 0,\ p \in \mathcal{P}\}$.

For finite nonnegative number δ, let $\mathcal{V}_\delta$ denote the class of all piecewise-continuous exogenous inputs $v:[0, \infty) \to \mathbb{R}^{n_v}$ for which $|v(t)| \le \delta$, $t \ge 0$. The preceding assumptions are sufficient to ensure that for each $v \in \mathcal{V}_\delta$ and each initial state $(x(0), x_N(0), k(0))$ in some bounded subset $\mathfrak{X} \subset \mathbb{R}^n \times \mathbb{R}^{n_N} \times \mathcal{P}$ there is an interval $[0, \bar{t})$ of maximal length on which a unique solution $(x(t), x_N(t), k(t))$ to (22)–(26) exists. Our objective is to show, under certain conditions, that $\tilde{t} = \infty$ and that such solutions are bounded on $[0, \infty)$. Quite often, the type of proof used to reach such conclusions involves two steps. The first often consists of using Lyapunov-like functions to establish the following properties.

Tuner Properties:

i) There exists a positive number C^*, depending only on $\mathfrak{X}$, δ, and $\bar{\mathcal{Q}}$ such that for each initial state in $\mathfrak{X}$, each exogenous input $v \in \mathcal{V}_\delta$ and each uncertainty vector $q \in \bar{\mathcal{Q}}$, the functions $|\dot{k}|$ and $|\bar{e}_T|$ are bounded by C^* along the solution to (22)–(26).

ii) There exists a continuous nonnegative function $\lambda^*: \mathcal{Q} \to \mathbb{R}$, depending only on $\mathfrak{X}$ and δ, with the property that for each initial state in $\mathfrak{X}$, each exogenous input $v \in \mathcal{V}_\delta$ and each uncertainty vector $q \in \bar{\mathcal{Q}}$, the functions $|\dot{k}|$ and $|W\bar{e}_T|$ are nondestabilizing along (x, x_N) with growth rates not exceeding $\lambda^*(q)$.

iii) $\lambda^*(0) = 0$.

The preceding properties are, in fact, easily established for many tuning algorithms discussed in the literature. They are, for example, characteristic of all algorithms discussed in [2]–[5], [10], [11], [20], [22], [31]. Note that properties ii) and iii) imply that the growth rates of $|\bar{e}_T|$ and $|W\bar{e}_T|$ will be smaller than any prescribed number $\epsilon > 0$, if the process model Σ_P to which the algorithm is being applied is sufficiently close to the nominal Σ_0 (i.e., $|q_P|$ is sufficiently small). On the other hand, with some algorithms, small growth rates may result, even when Σ_P is "very far" from Σ_0. For example, this is so with the classical algorithm of [2], where q_P is presumed to be a mismatch error, since in this case $\lambda^*(q) = 0$ for all $q \in \mathcal{Q}$.

As already mentioned, the first step in the analysis of (22)–(26) is to show that a system's tuner has the properties we have just enumerated. While considerable ingenuity is typically required to develop such algorithms, to verify that they possess these properties is usually straightforward. This is in sharp contrast to what is encountered in Step 2 where intricate and difficult arguments are often needed to prove that (x, x_N) is bounded. In the sequel, it will be shown that substantial simplification in Step 2 can be achieved by using the concept of an output injection [23]. To briefly illustrate the idea, suppose we want to prove for a particular initial condition that the state of a detectable system $y = Cx, \dot{x} = Ax$ is bounded (or goes to zero) given only that y is bounding (or zeroing). To do this, we use the fact that there is an (output injection) matrix H which exponentially stabilizes $A + HC$. This is a consequence of detectability. H need not be computed; it only has to exist. Now write $\dot{x} = (A + HC)x - Hy$. Since $A + HC$ is exponentially stable and y is bounding, we have, immediately, that x is bounded and if y is zeroing, then x goes to zero.

Our idea is to exploit the preceding in the more general context of a parameter adaptive control system. Since we will be dealing with matrix pairs $(C(p), A(p))$ which depend on $p \in \mathcal{P}$, we need some preliminary results.

Proposition 1: Let $\mathcal{Q}$ be an open subset of $\mathbb{R}^{n_q}$; let $\mathcal{E}$ and $\mathcal{P}$ be open subsets of $\mathbb{R}^{n_p}$ with $\mathcal{E} \subset \mathcal{P}$; let $(C, A): \mathcal{P} \to \mathbb{R}^{m \times n} \times \mathbb{R}^{n \times n}$ be a continuously differentiable matrix pair depending continuously on $q \in \mathcal{Q}$.

If $(C(p), A(p))$ is detectable for each $p \in \mathcal{E}$ and each $q \in \mathcal{Q}$, then the following hold.

i) The class $\mathcal{C}(C, A, \mathcal{E})$ of continuously differentiable matrices $H: \mathcal{E} \to \mathbb{R}^{n \times m}$, depending continuously on $q \in \mathcal{Q}$ for which $A(p) + H(p)C(p)$ is exponentially stable for each $p \in \mathcal{E}$ and each $q \in \mathcal{Q}$, is nonempty.

ii) For each matrix $H \in \mathcal{C}(C, A, \mathcal{E})$, there exists a unique symmetric, positive-definite, continuously differentiable matrix $R: \mathcal{E} \to \mathbb{R}^{n \times n}$, depending continuously on $q \in \mathcal{Q}$, which satisfies the Lyapunov equation

$$R(p)(A(p) + H(p)C(p)) + (A(p) + H(p)C(p))'R(p) + I = 0$$

for each $p \in \mathcal{P}$ and each $q \in \mathcal{Q}$.

iii) For each matrix $H \in \mathcal{C}(C, A, \mathcal{E})$ and each pair of compact subsets $\bar{\mathcal{E}}$ and $\bar{\mathcal{Q}}$ of $\mathcal{E}$ and $\mathcal{Q}$, respectively, there exists positive constants C and λ depending only on H, $\bar{\mathcal{E}}$ and $\bar{\mathcal{Q}}$ with the property that, for each continuously differentiable function $\zeta:[0, \tilde{t}) \to \bar{\mathcal{E}}$ and each $q \in \bar{\mathcal{Q}}$, the state transition matrix $\phi(t, \tau)$ of $A(\zeta(t)) + H(\zeta(t))C(\zeta(t)) - \frac{1}{2}R^{-1}(\zeta(t))\dot{R}(\zeta(t))$ satisfies

$$|\phi(t, \tau)| \leq Ce^{-\lambda(t-\tau)} \qquad 0 \leq \tau \leq t \leq \tilde{t}. \tag{28}$$

The last statement of this proposition is a slight generalization of [22, Lemma 6.2] (see also [20, Lemma 3]). The proposition's proof depends on the following lemma, which has been proved previously in [21].

Lemma 2: Let Ω be an open subset of all detectable matrix pairs $(C, A) \in \mathbb{R}^{m \times n} \oplus \mathbb{R}^{n \times n}$ and let S^n denote the linear space of all $n \times n$ symmetric matrices. There exists a unique analytic function $\bar{R}: \Omega \to S^n$ whose value $R = \bar{R}(C, A)$ at (C, A) is positive definite, exponentially stabilizes the matrix $A - RC'C$, and satisfies the matrix Riccati equation

$$AR + RA' - RC'CR + I = 0.$$

We now state our main result.

Tuning Theorem: Let (22)–(26) describe an adaptive control system consisting of a parameterized subsystem $\Sigma(p)$ and a tuner Σ_T. Suppose that for some compact subset $\bar{\mathcal{Q}} \subset \mathcal{Q}$ with dense interior containing the zero vector, $\Sigma(p)$ satisfies the Tunability Assumption. Suppose, in addition, for some finite number $\delta > 0$ and some bounded subset $\mathfrak{X}$ of the state space of (22)–(26), that Σ_T has Tuner Properties i) to iii). Then the following are true.

1) There exists a subset $\mathcal{Q}^* \subset \bar{\mathcal{Q}}$ with dense interior containing the zero vector, such that for each uncertainty vector $q \in \mathcal{Q}^*$, each initial state in $\mathfrak{X}$ and each exogenous input $v \in \mathcal{V}_\delta$, the solution $(x(t), x_N(t), k(t))$ to (22)–(26) exists and is bounded on $[0, \infty)$.

2) Suppose, in addition, that along a solution, $W\bar{e}_T$ has a bounded derivative and as $t \to \infty$, $\bar{e}_T$ and k approach limits 0 and k^*, respectively. Then $k^* \in \mathcal{E}_T$; moreover for each output injection matrix $H^* \in \mathcal{C}(C, A, \mathcal{E}_T)$, x approaches the unique bounded solution x^* the equation

$$\dot{x}^* = A(k^*)x^* + B(k^*)v + H^*(k^*)e^*$$

$$e^* = C(k^*)x^* + D(k^*)v$$

$$x^*(0) = 0,$$

and e^* approaches zero.

The Tuning Theorem's first statement implies that adaptive controllers satisfying the theorem's hypotheses are capable of stabilizing all process models $\Sigma_p \in \mathfrak{M}$ in some neighborhood of the nominal Σ_0. The additional hypotheses made in the theorem's second statement are often satisfied when the process model has no unmodeled dynamics and there are no external disturbances acting on the process (i.e., when v is just a reference input).

The point of the Tuning Theorem is, of course, that it is algorithm independent. The theorem owes no allegiance to any particular parameterization or design philosophy (e.g., direct or indirect control) leading to $\Sigma(k)$ nor to any one tuning algorithm Σ_T. All that is required is that $\Sigma(k)$ be tunable on $\mathcal{E}_T$ and that Σ_T possess Tuner Properties i)–iii). Tunability is just a bit stronger than weak tunability which, in turn, is necessary for stability and, as mentioned before, many tuners have the aforementioned properties. What this line of reasoning does then, is to bring into sharp focus those key features common to a large number of seemingly different adaptive control algorithms which are needed to verify their ability to stabilize.

Let $C_T: \mathcal{P} \to \mathbb{R}^{nn_p \times n}$ denote the matrix-valued function whose value at p is the block diagonal matrix consisting of n diagonal blocks each equal to $\Pi_T(p)$. Note that C_T is continuously differentiable since Π_T is. The following lemma is particularly useful.

Tunability Lemma: The matrix pair $(C(p), A(p))$ is detectable on $\mathcal{E}_T = \{p: \Pi_T(p) = 0,\ p \in \mathcal{P}\}$ iff the matrix pair

$$\left(\begin{bmatrix} C(p) \\ C_T(p) \end{bmatrix}, A(p)\right)$$

is detectable on $\mathcal{P}$.

Proof: if

$$\left(\begin{bmatrix} C(p) \\ C_T(p) \end{bmatrix}, A(p)\right)$$

is detectable on $\mathcal{P}$, then it is detectable on $\mathcal{E}_T$ since $\mathcal{E}_T \subset \mathcal{P}$; but $C_T(p) = 0$ for $p \in \mathcal{E}_T$ so $(C(p), A(p))$ is detectable on $\mathcal{E}_T$.

Now assume $(C(p), A(p))$ is detectable on $\mathcal{E}_T$. Fix $p \in \mathcal{P}$ and suppose that (λ, x) is an eigenvalue-eigenvector pair of $A(p)$ for which $C(p)x = 0$ and $C_T(p)x = 0$. To complete the lemma's proof, it is enough to show that real part $\lambda < 0$.

Since x is an eigenvector, it cannot be zero. This, the definition of $C_T(p)$, and the hypothesis $C_T(p)x = 0$ imply that $\Pi_T(p) = 0$ and thus, that $p \in \mathcal{E}_T$. Since, by assumption, $(C(p), A(p))$ is detectable on $\mathcal{E}_T$, it follows that real part $\lambda < 0$. □

Construction of $\mathcal{Q}^*$: We now explain how to construct a subset $\mathcal{Q}^*$ of process model uncertainty vectors $q \in \mathcal{Q}$ for which the stability of (22)–(26) can be assured. We take as given, a bounded subset $\mathcal{X}$ of initial states of (22)–(26), a positive number δ bounding admissible exogenous input v, and a compact subset $\bar{\mathcal{Q}} \subset \mathcal{Q}$ with dense interior containing the zero vector. We presume that the tuner Σ_T (23)–(26) has Tuner Properties i)–iii) and that the Tunability Assumption holds. $\mathcal{Q}^*$ is constructed in five steps.

Step 1: With C^* as given by Tuner Property i), define $\bar{\mathcal{P}} = \{p: |p| \le C^*,\ p \in \mathcal{P}\}$. The preceding lemma together with the Tunability Assumption and Remark 1 imply that

$$\left(\begin{bmatrix} C(p) \\ C_T(p) \end{bmatrix}, A(p)\right)$$

is detectable on $\mathcal{P}$. Since $\mathcal{P}$ is open, by i) of Proposition 1,

$$\mathcal{C}\left(\begin{bmatrix} C \\ C_T \end{bmatrix}, A, \mathcal{P}\right)$$

is nonempty. Pick

$$[H, H_T] \in \mathcal{C}\left(\begin{bmatrix} C \\ C_T \end{bmatrix}, A, \mathcal{P}\right)$$

and let R be the corresponding solution to the Lyapunov equation

$$R(A + HC + H_TC_T) + (A + HC + H_TC_T)'R + I = 0$$

as in ii) of Proposition 1.

Step 2: Let Ω denote the class of functions $\omega:[0, t_\omega) \to \bar{\mathcal{P}}$, with piecewise-continuous first derivatives and with t_ω specifying ω's interval of definition. For each $\omega \in \Omega$, and each $q \in \bar{\mathcal{Q}}$ define

$$U_\omega(t) = \frac{1}{2}R^{-1}(\omega(t))\dot{R}(\omega(t)) \tag{29}$$

and

$$\bar{A}_\omega(t) = A(\omega(t)) + H(\omega(t))C(\omega(t)) + H_T(\omega(t))C_T(\omega(t)) - U_\omega(t). \tag{30}$$

From the definition of U_ω and the properties of R given in ii) of Proposition 1, it follows that there is a nonnegative constant C_U, depending only on $\bar{\mathcal{Q}}$ and C^* such that

$$|U_\omega(t)| \le C_U|\dot{\omega}(t)|, \qquad q \in \bar{\mathcal{Q}},\ \omega \in \Omega,\ t \in [0, t_\omega). \tag{31}$$

By iii) of Proposition 1, there are positive constants C_1 and λ_1, depending only on $\bar{\mathcal{Q}}$ and C^*, H, and H_T, such that for each $\omega \in \Omega$, the state transition matrix $\phi_\omega(t, \tau)$ of $\bar{A}_\omega$ satisfies $|\phi_\omega(t, \tau)| \le C_1e^{-\lambda_1(t-\tau)}$, $0 \le \tau \le t \le t_\omega$. By hypothesis, A_N is exponentially stable so there also exist positive constants C_N and λ_N such that A_N's state transition matrix $\phi_N(t, \tau)$ satisfies $|\phi_N(t, \tau)| \le C_Ne^{-\lambda_N(t-\tau)}$, $0 \le \tau \le t \le t_\omega$. Define $C = \max\{C_1, C_N\}$ and $\lambda = \min\{\lambda_1, \lambda_N\}$; then

$$\left.\begin{aligned} |\phi_\omega(t, \tau)| &\le Ce^{-\lambda(t-\tau)} \\ |\phi_N(t, \tau)| &\le Ce^{-\lambda(t-\tau)} \end{aligned}\right\} \quad q \in \bar{\mathcal{Q}},\ \omega \in \Omega,\ 0 \le \tau \le t \le t_\omega. \tag{32}$$

Step 3: Define

$$C_S = (1 + |M|)nC_{H_T} \tag{33}$$

where C_{H_T} is the supremum over $\bar{\mathcal{Q}} \times \bar{\mathcal{P}}$ of $|H_T|$. Note that the definition of C_T, which appears just above the Tunability Lemma, implies that

$$|H_T(p)C_T(p)| \le nC_{H_T}|\Pi(p)|, \qquad q \in \bar{\mathcal{Q}},\ p \in \bar{\mathcal{P}}. \tag{34}$$

Step 4: Define $F_1: \mathcal{P} \times \mathbb{R}^n \times \mathbb{R}^{n_N} \times \mathbb{R}^{n_v} \to \mathbb{R}^{n \times m}$ and $F_2: \mathcal{P} \times \mathbb{R}^n \times \mathbb{R}^{n_v} \to \mathbb{R}^{n_N}$ so that for $p \in \mathcal{P}$, $x \in \mathbb{R}^n$, $x_N \in \mathbb{R}^{n_N}$, and $v \in \mathbb{R}^{n_v}$

$$F_1(p, x, x_N, v) = -H(p)W'(p, \bar{d}, x_N)N$$
$$F_2(p, x, v) = b_N(p, \bar{d}) \tag{35}$$

where

$$\bar{d}(p, x, v) = E(p)x + G(p)v. \tag{36}$$

In view of the linear growth assumptions (27) and the definition of $\bar{d}$ in (36) it follows that for $p \in \bar{\mathcal{P}}$ and $q \in \bar{\mathcal{Q}}$:

$$|F_1(p, x, x_N, v)| \le C_H(C_{W1}(C_E|x| + C_G|v| + |x_N|) + C_{W2})|N|$$
$$|F_2(p, x, v)| \le C_{N1}(C_E|x| + C_G|v|) + C_{N2}$$

where for $i = 1, 2$, C_{Wi} and C_{Ni} are growth constants depending only on $\bar{\mathcal{Q}}$ and $\bar{\mathcal{P}}$, and C_H, C_E, C_G are the suprema over $\bar{\mathcal{Q}} \times \bar{\mathcal{P}}$ of $|H|$, $|E|$, and $|G|$, respectively. Define $C_1 = C_{N1}C_E + 1$ and $C_F = C_HC_{W1}(C_E + 1)|N|$. It follows that for all x, x_N, v, $q \in \bar{\mathcal{Q}}$, and $p \in \bar{\mathcal{P}}$

$$\begin{aligned} |F_1| &\le C_F(|x| + C_1|x_N|) + C_FC_G|v| + C_HC_{W2}|N| \\ |F_2| &\le C_1|x| + C_{N1}C_G|v| + C_{N2}. \end{aligned} \tag{37}$$

Step 5: With $\lambda^*(q)$ as given by Tuner Property ii), define

$$\mathcal{Q}^* = \{q : (C_U + C_S + C_F)\lambda^*(q) + 2C_1\sqrt{(C_U + C_S + C_F)\lambda^*(q)} < \lambda/C,\ q \in \bar{\mathcal{Q}}\}. \tag{38}$$

The continuity of λ^* on $\mathcal{Q}$ [Tuner Property ii)], the presumption that $\lambda^*(0) = 0$ [Tuner Property iii)], and the hypothesis that $\bar{\mathcal{Q}}$ has a dense interior containing the zero vector, together imply that $\mathcal{Q}^*$ is nonempty, also with a dense interior containing the zero vector.

Remark 2: As constructed, $\mathcal{Q}^*$ depends on $\bar{\mathcal{Q}}$, C^*, $\bar{\mathcal{P}}$, H_T, and H; $\bar{\mathcal{P}}$, in turn, is a function of $\mathcal{Q}$ and C^*. Therefore, $\mathcal{Q}^*$ ultimately depends on just $\bar{\mathcal{Q}}$, C^*, H_T, and H. It would be useful to express this dependence in more explicit terms since, for fixed $\bar{\mathcal{Q}}$, H_T, and H, this would in effect give bounds on allowable process model uncertainty in terms of C^*. For most tuning algorithms, C^* is determined by $\mathfrak{X}$ and in some cases, by both $\mathfrak{X}$ and δ. It should, therefore, be possible to compute bounds on process model uncertainty, in terms of $\mathfrak{X}$ and δ. It would be interesting to see how these bounds depend on the choice of

$$[H, H_T] \in \left(\mathcal{C}\begin{bmatrix} C \\ C_T \end{bmatrix}, A, \mathcal{P}\right).$$

Proof of Tuning Theorem: It will first be shown that Assertion 1 of the theorem is true. For this, fix $q^* \in \mathcal{Q}^*$, $v \in \mathcal{V}_\delta$, initial state $(x(0), x_N(0), k(0)) \in \mathfrak{X}$ and let $(x(t), x_N(t), k(t))$ be the resulting solution of (22)–(26) with maximal interval of existence $[0, \bar{t})$. From Tuner Property i) and the definitions of $\bar{\mathcal{P}}$, it follows that $k(t) \in \bar{\mathcal{P}}$ for $t \in [0, \bar{t})$.

With U, $\bar{A}$, F_1, and F_2 given by (29), (30), (35), and (36) define A_1, A_2, f_1, f_2, and b along solution (x, x_N, k) so that for $t \in [0, \bar{t})$, $y \in \mathbb{R}^n$ and $y_N \in \mathbb{R}^{n_N}$

$$A_1(t) = \bar{A}_k$$

$$A_2(t) = A_N$$

$$f_1(y, y_N, t) = (U_k - H_T C_T)y - F_1(k, y, y_N, v)W\bar{e}_T$$

$$f_2(y, t) = F_2(k, y, v)$$

$$b = (B + HD)v - H\bar{e}_T. \tag{39}$$

Using these definitions together with (22) and (24), it is possible to write the differential equations for x and x_N in (22) and (26), respectively, as

$$\dot{x} = A_1(t)x + f_1(x, x_N, t) + b(t)$$

$$\dot{x}_N = A_2(t)x_N + f_2(x, t). \tag{40}$$

Next, observe that the definitions of f_1 and f_2 in (39) together with inequalities (31), (34), and (37) imply that

$$|f_1(y, y_N, t)| \le (C_U|\dot{k}| + nC_{H_T}|\Pi(k)|)|y| + (C_F(|y| + C_1|y_N|) + C_v)|W\bar{e}_T| \tag{41}$$

$$|f_2(y, t)| \le C_1|y| + C_2 \tag{42}$$

where $C_v = \delta C_F C_G + C_H C_{W_2}|N|$ and $C_2 = \delta C_{N1}C_G + C_{N2}$. From (23), $|\Pi(k)| \le |\dot{k}| + |M||W\bar{e}_T|$. Thus, (33) and (41) imply that

$$|f_1(y, y_N, t)| \le \sigma(|y| + C_1|y_N|) + h \tag{43}$$

where

$$\sigma = (C_U + C_S + C_F)\sup\{|\dot{k}|, |W\bar{e}_T|\}$$

$$h = |W\bar{e}_T|C_v. \tag{44}$$

Tuner Property i) together with the definition of b in (39), imply that b is bounded. In addition, Tuner Properties i) and ii) ensure that h is bounding and that σ is nondestabilizing along (x, x_N) with growth rate $\lambda^{**} \le (C_F + C_S + C_U)\lambda^*(q^*)$. But $q^* \in \mathcal{Q}^*$. In view of the definition of $\mathcal{Q}^*$ in (38), it must be true that $\lambda^{**} + 2C_1\sqrt{\lambda^{**}} < \lambda/C$. Therefore, by applying Lemma 1 to (40) with $\bar{t} = \bar{t}$, we conclude that $(x(t), x_N(t))$ is bounded on $[0, \bar{t})$. Thus, $(x(t), x_N(t), k(t))$ is bounded wherever it exists, so this solution must exist and be bounded on $[0, \infty)$.

Now, suppose that the hypotheses of the theorem's second statement are satisfied. Then the integral $\int_0^\infty \dot{k}\,dt$ converges and from (23), $\ddot{k}$ is bounded. These properties imply that $\dot{k} \to 0$. Since by assumption $\bar{e}_T \to 0$, it follows from (23) that $\Pi(k) \to 0$ as $k \to k^*$ or that $k^* \in \mathcal{E}_T$.

The preceding and (44) imply that h and σ are zeroing functions. Therefore, by Lemma 1, $x \to \bar{x}$ where

$$\dot{\bar{x}} = A_1(t)\bar{x} + b. \tag{45}$$

Let x^* and H^* be as in the tuning theorem's second statement and define $z = x - x^*$. From (30), (39), and (43) it follows that:

$$\dot{z} = (A(k^*) + H^*(k^*)C(k^*))z + \zeta \tag{46}$$

where

$$\begin{aligned}\zeta = {} & (A(k) - A(k^*))\bar{x} + H_T(k)C_T(k)\bar{x} - U_k\bar{x} \\ & + H^*(k^*)(C(k) - C(k^*))\bar{x} + (H(k) - H^*(k^*))C\bar{x} \\ & + (B(k) - B(k^*))v - H(k)\bar{e}_T + (H(k)D(k) \\ & - H^*(k^*)D(k^*))v.\end{aligned}$$

Since v and $\bar{x}$ are bounded, $k \to k^*$ and $\dot{k}$ and $\bar{e}_t$ go to 0, it follows that $C_T(k) \to C_T(k^*) = 0$, that $U_k \to 0$ because of (31), and thus, that $\zeta \to 0$. But $A(k^*) + H^*(k^*)C(k^*)$ is exponentially stable, so from (46), $z \to 0$. Therefore, $x \to x^*$ as claimed. This together with (22), (24), and the definition of e^* in the tuning theorem, imply that $e^* \to 0$. □

V. Conclusions

The purpose of this paper has been to illustrate some of the advantages of thinking of a parameter adaptive control system as a system consisting of a process, a parameterized controller, and a tuner, interconnected in a particular way. The proposed structure has the virtue of being general enough to describe many different kinds of adaptive systems including those of the model reference, self-tuning, and high-gain feedback types. While error models are not used in this setting, special emphasis is placed on the importance of a tuning error. This leads, naturally, to the concept of weak tunability which proves to be a fundamental property any parameter adaptive control system of the aforementioned type must have if stability is to be assured.

The tuning theorem of Section IV is applicable to a large class of adaptive control systems. By modifying the theorem's hypotheses, it should be possible to obtain new theorems appropriate to other classes of adaptive systems. For one such class, this has already been done. It has been shown in [15] that if the parameterized system $\Sigma(p) = (C(p), A(p))$: i) depends rationally and continuously on a scalar parameter $p \in \mathcal{P} = \mathbb{R}$; ii) is tunable on $\mathcal{P}'$, and iii) is "uniformly high-gain stabilized," then the state (x, k) of the adaptive system $\dot{k} = \|C(k)x\|^2$, $\dot{x} = A(k)x$ is bounded on $[0, \infty)$ and $x \to 0$ as $t \to \infty$.

One consequence of the ideas in this paper has been the realization that strict adherence to the Certainty-Equivalence Principle of indirect control is unnecessarily limiting. In a sequel to this paper [28], it is shown that *only* by discarding one of the principle's main tenets—pick feedback gains to stabilize the design model—it is possible to obtain with indirect control, algorithms with capabilities comparable to the classical algorithms of direct control.

Appendix

The proof of Lemma 1 depends on the following result which is a slight generalization of the Bellman–Gronwall Lemma.

Lemma 0: If for some constant $C \geq 0$ and nonnegative piecewise-continuous functions $\alpha:[t_1, t_2) \to \mathbb{R}$ and $\beta:[t_1, t_2) \to \mathbb{R}$, $u:[t_1, t_2) \to \mathbb{R}$ is a continuous function satisfying

$$u(t) \leq C + \int_{t_1}^{t} (\alpha(\tau)u(\tau) + \beta(\tau))\, d\tau, \qquad t \in [t_1, t_2) \quad \text{(A.1)}$$

then

$$u(t) \leq Ce^{\int_{t_1}^{t} \alpha(\tau)\, d\tau} + \int_{t_1}^{t} e^{\int_{\tau}^{t} \alpha(\mu)\, d\mu} \beta(\tau)\, d\tau, \qquad t \in [t_1, t_2). \quad \text{(A.2)}$$

Proof: Set

$$v(t) = u(t)e^{-\int_{t_1}^{t} \alpha(\tau)\, d\tau},$$

$$w(t) = \left(C + \int_{t_1}^{t} (\alpha(\tau)u(\tau) + \beta(\tau))\, d\tau\right) e^{-\int_{t_1}^{t} \alpha(\tau)\, d\tau}$$

and note from (A.1) that

$$v(t) \leq w(t), \qquad t \in [t_1, t_2). \quad \text{(A.3)}$$

Differentiating the expression for w and then replacing $u(t)$ by

$$v(t)e^{\int_{t_1}^{t} \alpha(\tau)\, d\tau}$$

gives

$$\dot{w} = \alpha(v - w) + \beta e^{-\int_{t_1}^{t} \alpha(\tau)\, d\tau}.$$

Hence, by (A.3),

$$\dot{w} \leq \beta e^{-\int_{t_1}^{t} \alpha(\tau)\, d\tau}.$$

Integrating this inequality and then using (A.3) we obtain

$$v(t) \leq C + \int_{t_1}^{t} e^{-\int_{t_1}^{t} \alpha(\mu)\, d\mu} \beta(\tau)\, d\tau, \qquad t \in [t_1, t_2).$$

Multiplying through by $e^{\int_{t_1}^{t} \alpha(\tau)\, d\tau}$ and then replacing $e^{\int_{t_1}^{t} \alpha(\tau)\, d\tau} v(t)$ by $u(t)$ yields (A.2), which is the desired result. □

Proof of Lemma 1: By hypothesis, $\lambda^* + 2C_1\sqrt{\lambda^*} < \lambda/C$. Thus, there exists a number $\bar{\lambda}$ satisfying

$$\bar{\lambda} < \lambda/C \quad \text{(A.4)}$$

and $\bar{\lambda} < \lambda^* + 2C_1\sqrt{\lambda^*}$. Therefore, since the number $\delta = -C_1 + \sqrt{C_1^2 + \lambda}$ satisfies

$$\delta^2 + 2C_1\delta = \lambda \quad \text{(A.5)}$$

there follows

$$\delta^2 > \lambda^*. \quad \text{(A.6)}$$

To prove that $y(t)$ is bounded on $[0, \bar{t})$, it is enough to show that

$$\bar{y}(t) = [y_1', (t), \delta y_2'(t)]' \quad \text{(A.7)}$$

exists and is bounded on $[0, \bar{t})$. For this, first define

$$\bar{\sigma} = \delta C_1 + (1 + C_1/\delta)\sigma. \quad \text{(A.8)}$$

In view of (A.6) and the hypothesis that σ is nondestabilizing along y with growth rate λ^*, there must be constants C_3 and C_4 such that

$$\int_{\tau}^{t} |\sigma(\mu)|\, d\mu \leq C_3 + \delta^2(t - \tau) + \int_{\tau}^{t} \frac{C_4}{1 + |y(\mu)|}\, d\mu, \qquad 0 \leq \tau \leq t \leq \tilde{t}.$$

This, together with (A.5) and the inequality $1 + |\bar{y}| \geq (1 + \delta)(1 + |y|)$ imply that $\bar{\sigma}$ satisfies

$$\int_{\tau}^{t} |\bar{\sigma}(\mu)|\, d\mu \leq \bar{C}_3 + \bar{\lambda}(t - \tau) + \int_{\tau}^{t} \frac{\bar{C}_4}{1 + |\bar{y}(\mu)|}\, d\mu, \qquad 0 \leq \tau \leq t \leq \tilde{t} \quad \text{(A.9)}$$

where $\bar{C}_3 = (1 + C_1/\delta)C_3$ and $\bar{C}_4 = (1 + \delta)(1 + C_1/\delta)C_4$.

By the variation of constants formula

$$\begin{aligned} y_1(t) &= \phi_{A_1}(t, t_1)y_1(t_1) + \int_{t_1}^{t} \phi_{A_1}(t, \tau)(f_1(y_1(\tau), y_2(\tau), \tau) + b(\tau))\, d\tau \\ y_2(t) &= \phi_{A_2}(t, t_1)y_2(t_1) + \int_{t_1}^{t} \phi_{A_2}(t, \tau)f_2(y_1(\tau), \tau)\, d\tau. \end{aligned} \quad \text{(A.10)}$$

From these equations and (6) it follows that:

$$|y_1(t)| \leq Ce^{-\lambda(t-t_1)}(|y_1(t_1)| + \int_{t_1}^{t} e^{-(t_1-\tau)}(|\sigma(\tau)|(|y_1(\tau)| + C_1|y_2(\tau)|) + |h(\tau)| + |b(\tau)|\, d\tau)$$

$$|y_2(t)| \leq Ce^{-\lambda(t-t_1)} \left(|y_2(\tau)| + \int_{t_1}^{t} e^{-(t_1-\tau)}(C_1|y_1(\tau)| + C_2)\, d\tau\right).$$

These inequalities, together with the definitions of $\bar{y}$ and $\bar{\sigma}$ in (A.7) and (A.8), yield

$$|\bar{y}(t)| \leq Ce^{-\lambda(t-t_1)} \left(|\bar{y}(t_1)| + \int_{t_1}^{t} e^{-(t_1-\tau)}(|\bar{\sigma}(\tau)|\, |\bar{y}(\tau)| + |h(\tau)| + |\bar{b}(\tau)|)\, d\tau\right)$$

where $\bar{b}(\tau) = |b(\tau)| + \delta C_2$. Multiplying both sides of the preceding by $e^{\lambda(t-t_1)}$ and then using Lemma 0, we obtain

$$e^{\lambda(t-t_1)}|\bar{y}(t)| \leq C|\bar{y}(t_1)|e^{\int_{t_1}^{t} C|\bar{\sigma}(\tau)|\, d\tau} + C\int_{t_1}^{t} e^{\int_{\tau}^{t} C|\bar{\sigma}(\mu)|\, d\mu} e^{\lambda(\tau-t_1)}(|h(\tau)| + |\bar{b}(t)|)\, dt.$$

Multiplying through by $e^{-\lambda(t-t_1)}$ and using (A.9) yields

$$|\bar{y}(t)| \leq Ce^{C\bar{C}_3}|\bar{y}(t_1)|e^{-\int_{t_1}^{t} \pi(\mu)\, d\mu} + Ce^{C\bar{C}_3}\int_{t_1}^{t} e^{-\int_{\tau}^{t} \pi(\mu)\, d\mu}(|h(\tau)| + |\bar{b}(\tau)|)\, d\tau \quad \text{(A.11)}$$

where

$$\pi(t) = \lambda - C\bar{\lambda} - C\bar{C}_4/(1+|\bar{y}(t)|). \quad (A.12)$$

In view of (A.4) there exists a positive number $\tilde{\lambda} < \lambda - C\bar{\lambda}$. Define

$$C_5 = \int_0^{\bar{t}} e^{-\tilde{\lambda}(t-\tau)}(|h(\tau)| + |\bar{b}(\tau)|)\,d\tau.$$

Since $|h|$ and $|\bar{b}|$ are bounding, $C_5 < \infty$.

Suppose $|\bar{y}|$ is not bounded on $[0, \bar{t})$. This and the continuity of $|\bar{y}(t)|$ imply that for any number $C_6 > 0$ there must exist a closed interval $|t_1, t_2| \subset [0, \bar{t})$ such that

$$|\bar{y}(t_1)| = |\bar{y}(0)| + C\bar{C}_4/(\lambda - C\bar{\lambda} - \tilde{\lambda}) \quad (A.13)$$

$$|\bar{y}(t)| \geq C\bar{C}_4/(\lambda - C\bar{\lambda} - \tilde{\lambda}), \qquad t \in [t_1, t_2] \quad (A.14)$$

$$|\bar{y}(t_2)| > C_6. \quad (A.15)$$

But (A.12) and (A.14) imply that $\pi(t) \geq \tilde{\lambda}$, $t \in [t_1, t_2]$. From this, (A.13), and (A.11) it follows that for $t \in [t_1, t_2]$

$$|\bar{y}(t)| \leq \mathrm{C}e^{C\bar{C}_3}|\bar{y}(0)| + C\bar{C}_4/(\lambda - C\bar{\lambda} - \tilde{\lambda}) + C_5.$$

Choosing C_6 equal to the right side of this inequality and evaluating the inequality at $t = t_2$, leads to a contradiction of (A.15). Thus, $\bar{y}(t)$ is bounded wherever it exists; in view of (A.7) this must also be true of y, so y must exist and be bounded on $[0, \bar{t})$.

To show that $y \to x^*$ when σ and h are zeroing functions, it is enough to show that $e = y - x^*$ goes to zero as $t \to \infty$. Since x^* satisfies (7), and y satisfies (5), from the variation of constants formula

$$e(t) = \phi_{A_1}(t, 0)y_1(0) + \int_0^t \phi_{A_1}(t, \tau)f_1(y(\tau), \tau)\,d\tau.$$

From this and (6)

$$|e_1(t)| \leq Ce^{-\lambda t}|y_1(0)| + C\int_0^t e^{-\lambda(t-\tau)}\beta(\tau)\,d\tau \quad (A.16)$$

where $\beta(t) = |\sigma(t)|(|y_1(t)| + C_1|y_2(t)|) + h(t)$. Note that β is a zeroing function, since $|y|$ is bounded and, by hypothesis, $|\sigma|$ and $|h|$ are zeroing functions. From this and (A.16), it follows that $e(t) \to 0$ as $t \to \infty$. □

Proof of Lemma 2: Let $F:\mathbb{R}^{m\times n} \oplus \mathbb{R}^{n\times u} \oplus S \to S$ denote the analytic function

$$F(C, A, M) = AM + MA' - MC'CM + I. \quad (A.17)$$

Fix $(C, A) \in \Omega$. Since (C, A) is detectable (i.e., (A', C') is stabilizable), there must exist a unique symmetric solution $\bar{R}$ to the Riccati equation $F(C, A, \bar{R}) = 0$ (cf. [26, ch. 12]). With $\bar{R}$ defining the value of $R(\cdot)$ at (C, A), $R(\cdot)$ is well defined and unique. In addition, by the implicit function theorem (cf. [17, p. 273]), $R(\cdot)$ will be analytic provided that at each point $(C, A) \in \Omega$, the Jacobian of $\dfrac{\partial F}{\partial M}(C, A, M)|_{M=\bar{R}(C,A)}$ is nonzero. This in turn will be so if for each fixed $(C, A) \in \Omega$, the $\bar{n} = n(n+1)/2$ matrices

$$M_i = \left.\frac{\partial F\left(C, A\sum_{i=1}^{\bar{n}} x_i E_i\right)}{\partial x_i}\right|_{\sum_{i=1}^{\bar{n}} x_i E_i = R(C,A)}$$

are linearly independent, $\{E_1, \cdots, E_{\bar{n}}\}$ being a basis for S.

To show that this is so, first note from (A.17) that $M_i = L(E_i)$ where $L:S \to S$ is the linear function defined by

$$L(X) = (A - R(C, A)C'C)X + X(A - R(C, A)C'C)'.$$

Since $(A - R(C, A)C'C)$ is a stability matrix (cf. [26, ch. 12]), $L(\cdot)$ is an isomorphism.

Now let $\{\mu_1, \cdots, \mu_{\bar{n}}\}$ be any set of numbers for which $\sum_{i=1}^{\bar{n}} \mu_i M_i = 0$; then $L(\sum_{i=1}^{\bar{n}} \mu_i E_i) = 0$ and since $L(\cdot)$ is an isomorphism, $\sum_{i=1}^{\bar{n}} \mu_i E_i = 0$. But $\{E_1, \cdots, E_{\bar{n}}\}$ is an independent set, so $\mu_i = 0$, $i = 1, \cdots, \bar{n}$. This proves that the M_i are independent, that the implicit function is applicable, and thus that $R(\cdot)$ is analytic. □

Proof of Proposition 1: In view of Lemma 2, the matrix function $R:\mathcal{E} \to \mathbb{R}^{n\times n}$ defined by $R(p) = \bar{R}(C(p), A(p))$, depends continuously on $q \in \mathcal{Q}$, is continuously differentiable on $\mathcal{E}$, and stabilizes $A(p) - R(p)C'(p)C(p)$, for $p \in \mathcal{E}$, $q \in \mathcal{Q}$. It follows that $H(p) = -R(p)C'(p)$ has all of the properties required for i) to be true.

Let H be any matrix in $\mathcal{C}(C, A, \mathcal{E})$. Then the pair $(0, (A(p) + H(p)C(p))')$ is continuous on $\mathcal{Q}$ and continuously differentiable and detectable on $\mathcal{E}$. It follows from Lemma 2 that $R(p) = \bar{R}(0, (A(p) + H(p)C(p))')$ has all of the properties required for ii) to be true.

To prove iii), define

$$\lambda_1 = \inf_{(q,p)\in\bar{\mathcal{Q}}\times\bar{\mathcal{E}}} \underline{\lambda}(R(p)), \quad \lambda_2 = \sup_{(q,p)\in\bar{\mathcal{Q}}\times\bar{\mathcal{E}}} \bar{\lambda}(R(p))$$

where $\underline{\lambda}(\cdot)$ and $\bar{\lambda}(\cdot)$ denote minimal and maximal eigenvalue, respectively. Since R is continuous on $\mathcal{Q} \times \mathcal{E}$ and positive definite on the compact subset $\bar{\mathcal{Q}} \times \bar{\mathcal{E}}$, it follows that $0 < \lambda_1 \leq \lambda_2 < \infty$. Clearly

$$\lambda_1\|x\|^2 \leq x'R(p)x \leq \lambda_2\|x\|^2, \qquad x \in \mathbb{R}^n,\ q \in \bar{\mathcal{Q}},\ p \in \bar{\mathcal{E}}. \quad (A.18)$$

For $x \in \mathbb{R}^n$, define $y(t) = \phi(t, \tau)x$ and $V = y'(t)R(\zeta(t))y(t)$. In view of assertion ii), $\dot{V} = -\|y\|^2$; but from (A.18), $\|y\|^2 \geq (1/\lambda_2)V$, so $\dot{V} \leq -(1/\lambda_2)V$. Clearly, $V(t) \leq e^{-1/\lambda_2(t-\tau)}V(\tau)$. Therefore, by (A.18), $\lambda_1\|y(t)\|^2 \leq \lambda_2 e^{-1/\lambda_2(t-\tau)}\|y(\tau)\|^2$. Hence, $\|\phi(t, \tau)x\|^2 \leq (\lambda_2/\lambda_1)e^{-1/\lambda_2(t-\tau)}\|x\|^2$ or with $\lambda = 1/2\sqrt{\lambda_2}$, $\|\phi(t, \tau)\| \leq n\sqrt{\lambda_2/\lambda_1}e^{-\lambda(t-\tau)}$. From this, it follows that (28) holds with $C = n^2\sqrt{\lambda_2/\lambda_1}$. □

Acknowledgment

Many people contributed to this work. The author is grateful to G. Kreisselmeier, whose invitation to lecture at the Carl–Cranz Workshop on Adaptive Control in Oberpfaffenhofen, Germany, provided the stimulus for the development of the conceptual framework described in this paper. He is also grateful to G. C. Goodwin for calling his attention to a preprint of [22] which contains, along with many other useful ideas, the essential idea upon which the proof of statement iii) of Proposition 1 is based. It is a pleasure to thank J. W. Polderman for pointing out that Lemma 2 could be proved using the implicit function theorem. C. Byrnes has made the author aware of an earlier version of this lemma, published in [21]. The definition of tunability, originally defined to be what would now be called tunability on $\mathcal{P}$, was sharpened as a result of useful discussions with P. A. Ioannou. Many of the ideas of this paper were first aired before the author's colleagues at Yale, D. E. Koditschek and K. S. Narendra; the author has profited from their comments.

References

[1] A. Feuer and A. S. Morse, "Adaptive control of single-input single-output linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, no. 4, pp. 557–569, Aug. 1978.

[2] A. S. Morse, "Global stability of parameter-adaptive control systems,"

IEEE Trans. Automat. Contr., vol. AC-25, no. 3, pp. 433–439, June 1980.

[3] K. S. Narendra, Y. H. Lin, and L. S. Valavani, "Stable adaptive controller design—Part II: Proof of stability," *IEEE Trans. Automat. Contr.*, vol. AC-25, no. 3, pp. 440–448, June 1980.

[4] P. A. Ioannou and K. S. Tsakalis, "A robust direct adaptive controller," *IEEE Trans. Automat. Contr.*, vol. AC-31, no. 11, pp. 1033–1043, Nov. 1986.

[5] K. S. Narendra and A. M. Annaswamy, "A new adaptive law for robust adaptation without persistent excitation," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 2, pp. 134–145, Feb. 1987.

[6] R. D. Nussbaum, "Some remarks on a conjecture in parameter adaptive control," *Syst. Contr. Lett.*, vol. 3, pp. 243–246, 1983.

[7] B. Martensson, "The order of any stabilizing regulator is sufficient *a priori* information for adaptive stabilizing," *Syst. Contr. Lett.*, vol. 6, no. 2, pp. 87–91, 1985.

[8] A. S. Morse, "A three-dimensional universal controller for the adaptive stabilization of any strictly proper minimum-phase system with relative degree not exceeding two," *IEEE Trans. Automat. Contr.*, vol. AC-30, no. 12, pp. 1188–1191, Dec. 1985.

[9] J. C. Willems and C. I. Byrnes, "Global adaptive stabilization in the absence of information on the sign of the high-frequency gain" (Lecture Notes in Control and Information Sciences), in *Proc. 6th Int. Conf. Analysis Optimiz. Syst.*, Nice, France, June 1984, pp. 49–57.

[10] D. R. Mudgett and A. S. Morse, "Adaptive stabilization of linear systems with unknown high-frequency gains," *IEEE Trans. Automat. Contr.*, vol. AC-30, no. 6, pp. 549–554, June 1985.

[11] A. S. Morse, "A $4(n + 1)$-dimensional model reference adaptive stabilizer for any relative degree one or two, minimum phase system of dimension n or less," *Automatica*, vol. 23, no. 1, pp. 123–125, Jan. 1987.

[12] M. Fu and B. R. Barmish, "Adaptive stabilization of linear systems via switching control," *IEEE Trans. Automat. Contr.*, vol. AC-31, no. 12, pp. 1097–1103, Dec. 1986.

[13] A. S. Morse, "Simple algorithms for adaptive stabilization," in *Proc. ISSA Conf. Modelling Adaptive Contr.*, Sopron, Hungary, July 1986.

[14] ——, "High-gain feedback algorithms for adaptive stabilization," in *Proc. Yale Workshop Adaptive Contr.*, May 1987.

[15] ——, "Adaptive stabilization by high-gain feedback," submitted for publication.

[16] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. New York: Academic, 1975.

[17] J. K. Hale, *Ordinary Differential Equations*. New York: Interscience, 1969.

[18] J. Dieudonné, *Foundations of Modern Analysis*. New York: Academic, 1969.

[19] R. Bellman, *Stability Theory of Differential Equations*. New York: Dover, 1953.

[20] G. Kreisselmeier, "An approach to stable indirect adaptive control," *Automatica*, vol. 21, no. 4, pp. 425–431, 1985.

[21] D. F. Delchamps, "Analytic feedback control and the algebraic Riccati equation," *IEEE Trans. Automat. Contr.*, vol. AC-29, no. 11, pp. 1031–1033, Nov. 1984.

[22] R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mayne, "Design issues in adaptive control," *IEEE Trans. Automat. Contr.*, vol. 33, no. 1, pp. 50–58, Jan. 1988.

[23] A. S. Morse, "Structural invariance of linear multivariable systems," *SIAM J. Contr.*, vol. 11, no. 3, pp. 446–465, Aug. 1973.

[24] A. S. Morse and W. M. Wonham, "Status of non-interacting control," *IEEE Trans. Automat. Contr.*, vol. AC-16, no. 6, pp. 568–581, Dec. 1971.

[25] M. A. Duarte and K. S. Narendra, "Combined direct and indirect approach to adaptive control," Center for Syst. Sci., Yale Univ., New Haven, CT, Rep. 8711, Sept. 1987.

[26] W. M. Wonham, "Linear multivariable control—A geometric approach," in *Springer Lecture Notes in Economic and Mathematical Systems*. New York: Springer-Verlag, 1974.

[27] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete-time multivariable adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-25, no. 3, pp. 449–455, June 1980.

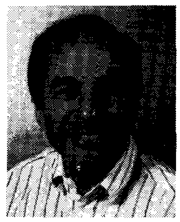
[28] A. S. Morse, "Towards a unified theory of parameter adaptive control—part 2: Certainty equivalence and implicit tuning," submitted for publication.

[29] D. E. Miller, "Adaptive control of uncertain systems," Ph.D. dissertation, Univ. Toronto, Oct. 1989.

[30] R. V. Monopoli, "Model reference adaptive control with an augmented error signal," *IEEE Trans. Automat. Contr.*, vol. AC-19, no. 4, pp. 474–484, Oct. 1974.

[31] P. Ioannou and J. Sun, "Theory and design of robust direct and indirect adaptive-control schemes," *Int. J. Contr.*, vol. 47, no. 3, pp. 775–813, 1988.

[32] L. Praly, "Global stability of direct adaptive control schemes with respect to a group topology," in *Adaptive and Learning Systems*. New York: Plenum, 1986, pp. 1009–1014.



A. Stephen Morse (S'62–M'67–F'84) was born in Mt. Vernon, NY, on June 18, 1939. He received the B.S.E.E. degree from Cornell University, Ithaca, NY, in 1962, the M.S. degree from the University of Arizona, Tucson, in 1964, and the Ph.D. degree from Purdue University, West Lafayette, IN, in 1967, all in electrical engineering.

From 1967 to 1970 he was associated with the Office of Control Theory and Application, NASA Electronics Research Center, Cambridge, MA. Since July 1970 he has been with Yale University, New Haven, CT, where he is a Professor of Electrical Engineering. His main interest is in system theory, and he has done research in network synthesis, optimal control, multivariable control, adaptive control, and urban transportation.

Dr. Morse is a member of Sigma Xi and Eta Kappa Nu. He has served as an Associate Editor of the IEEE Transactions on Automatic Control and as a Director of the American Automatic Control Council representing the Society for Industrial and Applied Mathematics.