1 Preliminaries on Continuity

Recall that we looked at a special class of operators last week known as contraction mappings. Given a closed subset M of a Banach space, an operator $T : \mathsf{M} \to \mathsf{M}$ is a contraction mapping if, for some $c \in [0, 1)$, $\|Tx - Tx'\| \leq c\|x - x'\|$ for all $x, x' \in \mathsf{M}$. Using this, we proved the Banach Fixed Point theorem. In this section, we will look at continuous mappings in Banach spaces, as well as the important related notion of relatively compact and compact sets.

Definition 1.1 *Let* $(\mathsf{X}, \|.\|_{\mathsf{X}})$ *and* $(\mathsf{Y}, \|.\|_{\mathsf{Y}})$ *be two Banach spaces. A mapping* $T : \mathsf{X} \to \mathsf{Y}$ *is continuous at* $x_0 \in \mathsf{X}$ *if, for all* $\varepsilon > 0$*, there exists a* $\delta = \delta(x_0, \varepsilon) > 0$ *such that* $\|x - x_0\|_{\mathsf{X}} < \delta$ *implies* $\|Tx - Tx_0\|_{\mathsf{Y}} < \varepsilon$*.*

Note that it is sometimes useful to consider the negation of the definition in order to prove certain results, as follows:

If $T : \mathsf{X} \to \mathsf{Y}$ is not continuous at x_0, then there exists an $\varepsilon_0 > 0$ such that for all $\delta > 0$ there exists some $x = x(x_0, \varepsilon)$ such that $\|x - x_0\|_{\mathsf{X}} < \delta$ but $\|Tx - Tx_0\|_{\mathsf{Y}} > \varepsilon$.

From now on, we will omit the subscripts X and Y on the norms; the underlying space should be clear from context. It is clear that all contraction mappings are continuous functions. For a given $\varepsilon > 0$, $\|x - x_0\| < \frac{\varepsilon}{c}$ implies $\|Tx - Tx_0\| \leq c\|x - x_0\| < \varepsilon$, where c is the contraction constant.

We can also define continuity on a set $\mathsf{M} \subseteq \mathsf{X}$ as follows.

Definition 1.2 *A mapping* $T : \mathsf{X} \to \mathsf{Y}$ *is continuous on* $\mathsf{M} \subseteq \mathsf{X}$ *if, for every* $\varepsilon > 0$ *and every* $x_0 \in \mathsf{M}$*, there exists a* $\delta = \delta(x_0, \varepsilon) > 0$ *such that* $x \in \mathsf{M}$ *with* $\|x - x_0\| < \delta$ *implies* $\|Tx - Tx_0\| < \varepsilon$*.*

A stricter notion of continuity called uniform continuity places a constraint that the function should converge at all points at the same speed.

Definition 1.3 *A mapping* $T : \mathsf{M} \subseteq \mathsf{X} \to \mathsf{Y}$ *is uniformly continuous if, for all* $\varepsilon > 0$*, there exists a* $\delta = \delta(\varepsilon) > 0$*, such that for all* $x, x' \in \mathsf{M}$*,* $\|x - x'\| < \delta$ *implies* $\|Tx - Tx_0\| < \varepsilon$*.*

Another notion of continuity that is equivalent to Def. 1.1 in normed spaces is sequential continuity:

Definition 1.4 *A mapping* $T : \mathsf{M} \subseteq \mathsf{X} \to \mathsf{Y}$ *is sequentially continuous at* $x_0 \in \mathsf{M}$ *if for any sequence* $(x_n)_{n\geq 1}$ *in* M *such that* $x_n \to x_0$*,* $Tx_n \to Tx_0$ *as* $n \to \infty$*.*

Recall from calculus that a real-valued function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $t \in \mathbb{R}$ iff $\lim_{n\to\infty} f(t_n) = f(t)$ for all $(t_n)_{n\geq 1}$ such that $\lim_{n\to\infty} t_n = t$. The above definition is a generalization of this to arbitrary normed spaces. We will now prove that sequential continuity is equivalent to continuity in normed spaces.

Theorem 1.1 *Given a mapping* $T : \mathsf{M} \subseteq \mathsf{X} \to \mathsf{Y}$*, the following are equivalent:*

1. *T is continuous*

2. *T is sequentially continuous.*

Proof: First we will prove that continuity implies sequential continuity ($1 \implies 2$). Suppose T is continuous at $x_0 \in \mathsf{M}$, and let $(x_n)_{n\geq 1}$ be a sequence converging to x_0 in M. By continuity, for any $\varepsilon > 0$ there exists some $\delta > 0$, such that $\|x - x_0\| < \delta$ in M implies $\|Tx - Tx_0\| < \varepsilon$. Since $x_n \to x_0$, there exists some $N \geq 1$, such that $\|x_n - x_0\| < \delta$ for all $n \geq N$. Hence, $\|Tx_n - Tx_0\| < \varepsilon$ for all $n \geq N$, which proves that $Tx_n \to Tx_0$ as $n \to \infty$.

We will now prove that sequential continuity implies continuity ($2 \implies 1$). Suppose that T is not continuous at x_0. Then there exists some $\varepsilon_0 > 0$ and, for each $n \in \mathbb{N}$, some $x_n \in \mathsf{M}$ with $\|x_n - x_0\| < 1/n$ and $\|Tx_n - Tx_0\| \geq \varepsilon_0$. But then the sequence $(x_n)_{n\geq 1}$ evidently converges to x_0 in M, and yet $Tx_n \not\to Tx_0$, which means that T is not sequentially continuous at x_0. ■

2 Compactness

We will now look at compactness in Banach spaces.

Definition 2.1 *Let M be a subset of a Banach space $(\mathsf{X}, \|.\|)$. Then M is:*

1. Relatively compact *if any sequence $(x_n) \in \mathsf{M}$ has a convergent subsequence. Note that the limit of the subsequence need not necessarily be in M.*

2. Compact *if any sequence $(x_n) \in \mathsf{M}$ has a convergent subsequence, with the limit contained in M.*

Remark 2.1 *Strictly speaking, the above definition introduces* sequential *compactness, but the distinction between sequential compactness and compactness, which matters in a general topological space, disappears in normed spaces.*

Lemma 2.1 *A set M is compact iff it is relatively compact and closed.*

Proof: Let M be compact, therefore relatively compact. Let (x_n) be a sequence in M that converges to some $x \in \mathsf{X}$. Since M is compact, there exists a convergent subsequence with limit $\bar{x} \in \mathsf{M}$. However, since all convergent subsequences of a convergent sequence converge to the same limit as the original sequence, we see that $x = \bar{x} \in \mathsf{M}$. Thus, M is relatively compact and closed.

Conversely, suppose M is closed and relatively compact. The latter implies that any sequence (x_n) in M has a convergent subsequence $(x_{n(i)})_{i\geq 1}$. Since M is closed, the limit of this subsequence lies in M. This implies that any sequence $(x_n) \in \mathsf{M}$ has a convergent subsequence with limit in M. Thus, M is compact. ■

Proposition 2.1 *Any relatively compact set is bounded.*

Proof: A set M is bounded iff there exists $r \in [0, \infty)$ such that $\|x\| \leq r$ for all $x \in \mathsf{M}$. Assume that M is relatively compact but not bounded. Then, for all $n = 1, 2, \ldots$, there exists $x_n \in \mathsf{M}$ such that $\|x_n\| \geq n$.

By relative compactness, this sequence (x_n) has a convergent subsequence $(x_{n(i)})_{i \geq 1}$. However, by construction, $\|x_{n(i)}\| \geq n(i)$ for all i. This contradicts the statement that $(x_{n(i)})_i$ has a limit. Thus, any relatively compact set is bounded. ■

This gives the immediate corollary:

Corollary 2.1 *Any compact set is bounded and closed.*

However, the converse implication holds only in finite-dimensional spaces. We will now see an example of a finite dimensional Banach space in which all bounded sets are relatively compact.

Proposition 2.2 *Consider the Banach space* $(\mathbb{R}^N, |.|_\infty)$. *Then any bounded set in this space is relatively compact.*

Proof: We will prove this by induction.

For $N = 1$, the statement follows from the Bolzano-Weirestrass theorem (any bounded sequence of reals has a convergent subequence).

For $N = 2$, consider the sequence $x_n = (\xi_{1n}, \xi_{2n})$ in M. Since M is bounded, there exists some $r < \infty$, such that

$$|\xi_{1n}|, |\xi_{2n}| \leq |x_n|_\infty \leq r, \qquad \forall n.$$

Since the sequence (ξ_{1n}) is bounded in $\mathbb{R}$, by the Bolzano-Weierstrass theorem it has a convergent subsequence $(\xi_{1n_1(i)})$ that converges to some $\xi_1 \in \mathbb{R}$. Now, the sequence $(\xi_{2n_1(i)})_{i \geq 1}$ is a bounded subsequence of $(\xi_{2n})_{n \geq 1}$, and therefore has a convergent subsequence $(\xi_{2n_2(i)})_{i \geq 1}$ with limit $\xi_2 \in \mathbb{R}$. Now, consider the sequence $(x_{\hat{n}}(i))_{i \geq 1}$ with $x_{\hat{n}(i)} := (\xi_{1n_2(i)}, \xi_{2n_2(i)})$. It follows from the above that this sequence is a convergent subsequence of (x_n), and has limit $x := (\xi_1, \xi_2)$. Thus, $(\mathbb{R}^2, |.|_\infty)$ is relatively compact.

A similar inductive argument works for all $N \geq 2$. ■

Next, we formalize the Diagonal Sequence argument, which may be seen as an extension of the argument made in the previous proposition. It comes in handy when proving results that require finding convergent subsequences, especially in infinite dimensional spaces.

Theorem 2.1 *Let* $f_n : \mathbb{N} \to \mathbb{R}$, $n \geq 1$, *be a bounded collection of functions, that is:*

$$\max_n \max_m |f_n(m)| < \infty.$$

Then there exists a subsequence $(f_{\hat{n}(k)})_{k \geq 1}$, *such that* $\lim_{k \to \infty} f_{\hat{n}(k)}(m)$ *exists for every* $m \in \mathbb{N}$.

Proof: Consider the bounded sequence $(f_n(1))_{n\geq 1}$. By the Bolzano-Weierstrass theorem, there exists a subsequence $(f_{n_1(i)}(1))_{i\geq 1}$ of $(f_n(1))_{n\geq 1}$, such that $\lim_{i\to\infty} f_{n_1(i)}(1)$ exists and is finite. We denote it by $f_\infty(1)$.

Next, we look at a bounded sequence for $m=2$, i.e., $(f_{n_1(i)}(2))_{i\geq 1}$. By the same argument as before, it has a subsequence $(f_{n_2(i)}(2))_{i\geq 1}$, such that $\lim_{i\to\infty} f_{n_2(i)}(2) = f_\infty(2)$ is also finite in $\mathbb{R}$. Proceeding inductively, we can construct the sequences $(f_{n_k(i)})_{i\geq 1}$ for each k with the following properties:

- $(f_{n_{k+1}(i)})_{i\geq 1}$ is a subsequence of $(f_{n_k(i)})_{i\geq 1}$;
- $f_\infty(k) = \lim_{i\to\infty} f_{n_k(i)}(k)$ exists and is finite.

This construction yields the following: for each k and each $1 \leq j \leq k$,

$$\lim_{i\to\infty} f_{n_k(i)}(j) = f_\infty(j)$$

(immediate for $j = k$; for $j < k$, this is a consequence of the fact that $(f_{n_k(i)}(j))_{i\geq 1}$ is a convergent subsequence of $(f_{n_j(i)}(j))_{i\geq 1}$, and thus converges to $f_\infty(j)$).

Consider now the 'diagonal' sequence $(f_{\hat{n}(k)})_{k\geq 1}$ with $\hat{n}(k) := n_k(k)$. This is evidently a subsequence of $(f_n)_{n\geq 1}$, and it remains to show that the limit $f_{\hat{n}(k)}(m)$ as $k\to\infty$ exists for each $m \in \mathbb{N}$. To that end, we first note that $f_{\hat{n}(m)} = f_{n_m(m)}, f_{\hat{n}(m+1)} = f_{n_{m+1}(m+1)}, \ldots$ is a subsequence of $(f_{n_m(k)})_{k\geq 1}$, and therefore $\lim_{k\to\infty} f_{\hat{n}(k)}(m) = \lim_{k\to\infty} f_{n_m(k)}(m) = f_\infty(m)$. ■

Next, we cover an important result in functional analysis for real-valued continuous functions defined on a closed and bounded interval, namely the Arzelà-Ascoli Theorem.

Theorem 2.2 ***(Arzelà-Ascoli)****: Consider the Banach space* $(\mathsf{X}, \|.\|) = (C[a,b], \|\cdot\|_\infty)$ *and let* $M \subseteq \mathsf{X}$ *satisfy the following properties:*

1. *boundedness:* $\sup_{x\in\mathsf{M}} \|x\|_\infty < \infty$*;*
2. *equicontinuity: for any* $\varepsilon > 0$*, there exists* $\delta > 0$*, such that* $|x(t) - x(s)| < \varepsilon$ *for all* $x \in \mathsf{M}$ *and all* $s, t \in [a,b]$ *with* $|t - s| < \delta$*.*

Then M is relatively compact

Proof: Let $(x_n)_{n\geq 1}$ be a sequence in M. Let $\widetilde{\mathbb{Q}} := \mathbb{Q} \cap [a,b]$, i.e., the set of all rationals in the interval $[a,b]$. Since $\widetilde{\mathbb{Q}}$ is countable, we can enumerate its elements in some way, i.e., $\widetilde{Q} = \{r_1, r_2, \ldots\}$.

Let $f_n : \mathbb{N} \to \mathbb{R}$ be a sequence of functions, such that $f_n(i) := x_n(r_i)$. Since M is bounded, it follows that

$$\max_n \max_i |f_n(i)| < \infty$$

Consequently, by the Diagonal Sequence argument, there exists a subsequence $(x_{\hat{n}(k)})_{k\geq 1}$, such that

$$\lim_{k\to\infty} x_{\hat{n}(k)}(r_i) = \lim_{k\to\infty} f_{\hat{n}(k)}(i)$$

exists for all i. Now define $y_k := x_{\hat{n}(k)}$ and observe that $y_k \in \mathsf{M}$ for all k. Moreover, (y_k) is a subsequence of x_n, and, for each i, the sequence $(y_k(r_i))_{k\geq 1}$ has a limit, by the above diagonalization argument.

We will now show that $(y_k)_{k\geq 1}$ has a limit, and (x_n) has a convergent subsequence. To that end, fix some $\varepsilon > 0$. By equicontinuity of M, there exists some $\delta > 0$, such that $|s - t| < \delta$ implies $|y_n(t) - y_n(s)| < \varepsilon$ for all n. Now, by the properties of rational numbers, there exist *finitely many* points $s_1, \ldots, s_J \in \widetilde{\mathbb{Q}}$, such that

$$\max_{t\in[a,b]} \min_{j\in[J]} |t - s_j| < \delta.$$

Then, for any $t \in [a, b]$ and any $m, n \geq 1$,

$$|y_m(t) - y_n(t)| \leq |y_m(t) - y_m(s_j)| + |y_m(s_j) - y_n(s_j)| + |y_n(s_j) - y_n(t)|,$$

where s_j satisfies $|t - s_j| = \min_{i\in\{J\}} |t - s_i| < \delta$, and we apply the triangle inequality after adding and subtracting terms.

By equicontinuity, $|y_m(t) - y_m(s_j)| < \varepsilon$ and $|y_n(t) - y_n(s_j)| < \varepsilon$ by the choice of s_j. Moreover, given the finitely many convergent sequences $(y_k(s_j))_{k\geq 1}$, $1 \leq j \leq J$, we can find some n_0, such that $\max_{j\in[J]} |y_m(s_j)-y_n(s_j)| < \varepsilon$ for all $m, n \geq n_0$. Consequently, we have that $\sup_{t\in[a,b]} |y_m(t)-y_n(t)| < 3\varepsilon$ for all $m, n \geq n_0$.

We have shown that, an arbitrary sequence $(f_n) \in \mathsf{M}$ has a subsequence (y_k) which is Cauchy and therefore converges, thus proving the relative compactness of M. ■

3 Total Boundedness

Next, we'll see how relatively compact sets are different from bounded sets. In the following discussion, $\mathsf{M} \subseteq (\mathsf{X}, \|.\|)$, a Banach space.

Definition 3.1 *A finite set* $\{x_1, x_2, ..., x_j\} \subseteq \mathsf{M}$ *is an* ε*-net if* $\forall x \in \mathsf{M}$, $\min\limits_{j\in[J]} \|x - x_j\| \leq \varepsilon$.

Definition 3.2 M *is totally bounded if it has a (finite)* ε*-net for every* $\varepsilon > 0$.

Theorem 3.1 M *is relatively compact iff it is totally bounded.*

Proof: First, we'll prove that if M is relatively compact, then it is totally bounded. Suppose M is not totally bounded. This means that there exists $\varepsilon_0 > 0$, such that for any finite set $\mathcal{S} \subset \mathsf{M}$ there exists some $x \in \mathsf{M}$ such that

$$\|x - y\| \geq \varepsilon_0, \forall y \in \mathcal{S}.$$

Now, fix $x_0 \in \mathsf{M}$ and let $\mathcal{S} = \{\S\}$. Since M is not totally bounded, there exists $x_1 \in \mathsf{M}$ such that $||x_1 - x_0|| > \varepsilon_0$. We now add this point x_1 to the set $\mathcal{S}$, so $\mathcal{S} = \{\S, \S\}$. Again, due to the fact that M is not totally bounded, there exists $x_2 \in \mathsf{M}$, such that $\|x_2 - x_0\| > \varepsilon_0, \|x_2 - x_1\| > \varepsilon_0$. We can keep repeating this argument to obtain a sequence $(x_n)_{n\geq 1}$ of points of M, such that $\|x_m - x_n\| > \varepsilon_0$ $\forall m, n$ with $m > n$. This sequence evidently has no convergent subsequence, so M is not relatively compact.

Next we'll prove that totally bounded implies relatively compact. We'll use the diagonal sequence argument to extract a convergent subsequence. Consider $(x_n)_{n\geq 1}$, an arbitrary sequence in M. Let $N_1 = \{y_1^{(1)}, ..., y_{J_1}^{(1)}\}$ be a 1-net. Notice that there exist infinitely many x_n's such that $\|x_n - y_j^{(1)}\|$ for some $j \in [J_1]$ (otherwise infinitely many x_n's can't be approximated by N_1, which contradicts the fact that N_1 is a 1-net). Let us denote this subsequence by $(x_n^{(1)})$. Then

$$\|x_m^{(1)} - x_n^{(1)}\| \leq \|x_m^{(1)} - y_j^{(1)}\| + \|x_n^{(1)} - y_j^{(1)}\| \leq 2.$$

Now consider N_2 as a $\frac{1}{2}$ net, $N_2 = \{y_1^{(2)}, ..., y_{J_2}^{(2)}\}$ and extract a subsequence $(x_n^{(2)})$ of $(x_n^{(1)})$ such that $\|x_m^{(2)} - x_n^{(2)}\| \leq \|x_m^{(2)} - y_j^{(2)}\| + \|x_n^{(2)} - y_j^{(2)}\| \leq 1$. Repeat this extraction of subsequences to obtain, for each k, a sequence $(x_n^{(k)})_{n\geq 1}$ such that $\|x_m^{(k)} - x_n^{(k)}\| \leq \frac{2}{k}$ $\forall m, n$ and $(x_n^{(k+1)})_{n\geq 1}$ is a subsequence of $(x_n^{(k)})_{n\geq 1}$. Denote $x_{\hat{n}(k)} := x_k^{(k)}$; by the diagonal sequence argument, the subsequence $(x_{\hat{n}(k)})$ of (x_n) is Cauchy. Since X is Banach, $(x_{\hat{n}})$ converges. Thus, M is relatively compact. ■

4 Weierstrass Theorem and Application

In this section, we'll see optimization of functionals over compact sets through Weierstrass theorem and its application to proving equivalence of norms on $\mathbb{R}^N$

Definition 4.1 *A function $f : \mathsf{X} \to \mathbb{R}$ is upper semicontinuous (u.s.c.) at $x_0 \in \mathsf{X}$ if $\forall \varepsilon > 0, \exists \delta$ such that $|x - x_0| < \delta \implies f(x) < f(x_0) + \varepsilon$*

Definition 4.2 *A function $f : \mathsf{X} \to \mathbb{R}$ is lower semicontinuous (l.s.c.) at $x_0 \in \mathsf{X}$ if $-f$ is upper semicontinuous at $x_0 \in \mathsf{X}$*

Theorem 4.1 (Weierstrass Theorem) *Let f be upper semicontinuous at every point of a compact set M; then f attains its (finite) maximum value on M.*

Note: An equivalent statement can be made about a l.s.c. function that it attains its minima in a compact set.

Proof: Let $a := \sup\limits_{x\in\mathsf{M}} f(x)$ (a can be ∞), then $\exists (x_n)$ in M such that $f(x_n) \to a$ (by definition of supremum). Now M is compact, thus $\exists (x_{n(i)})_{i=1,2,..}$ such that $x_{n(i)} \xrightarrow{i\to\infty} x_0$ in M. Also, f is u.s.c., thus $\forall \varepsilon > 0$, $\exists i_0(\varepsilon)$ such that $f(x_{n(i)}) < f(x_0) + \varepsilon$ $\forall n(i) \geq i_0\,(\varepsilon)$. Since $f(x_{n(i)})$ converges to a, $f(x_0) > a$. This contradicts the fact that a is supremum. So, $f(x_0) = a = \sup\limits_{x\in\mathsf{M}} f(x) < \infty$. ■

This theorem has numerous applications; one such application is to show the following:

Theorem 4.2 *All norms on $\mathbb{R}^N$ are equivalent*

Proof: Notice that the norm function $\|.\|$ is continuous as $|\|x\| - \|x'\|| \leq \|x - x'\|$. Consider a unit sphere in $(\mathbb{R}^N, |\cdot|_\infty)$, $\mathcal{S}_\infty := \{x \in \mathbb{R}^N : |x|_\infty = 1\}$. This is a compact set in $\mathbb{R}^N$ and $f(x) = \|x\|$ is a continuous function on this compact set. Thus, by the Weierstrass theorem, $\exists \underline{x}, \overline{x} \in \mathcal{S}_\infty$

such that $0 < \underline{x} \leq \|x\| \leq \overline{x}$. Now $\forall x \in \mathbb{R}^N \backslash \{0\}$, $\frac{x}{\|x\|} \in \mathcal{S}_\infty$. Thus, $\|\underline{x}\| \leq \|\frac{x}{|x|_\infty}\| \leq \|\overline{x}\|$. Thus, $\|\underline{x}\||x|_\infty \leq \|x\| \leq \|\overline{x}\||x|_\infty$.

We have proved that any norm $\|\cdot\|$ on $\mathbb{R}^N$ is equivalent to the ℓ^∞ norm $|\cdot|$. The corresponding statement about equivalence of any two arbitrary norms follows easily. ■

Next we give a few facts about Banach spaces before moving onto Hilbert spaces in the next lecture:

1. Example of a closed and bounded set that is not compact: Consider $(\mathsf{X}, \|\cdot\|) = (C[0,1], \|.\|_\infty)$ and the set $B := \{x \in C[0,1] : \|x\|_\infty \leq 1\}$ that is closed and bounded. Define $f : \mathsf{X} \to \mathbb{R}$ as $f(x) = \int_0^{\frac{1}{2}} x(t)\,\mathrm{d}t - \int_{\frac{1}{2}}^1 x(t)\,\mathrm{d}t$. f is continuous as $|f(x) - f(x')| \leq \|x - x'\|_\infty$. B is not compact as $\sup\limits_{x \in B} f(x) = 1$ but it can't be achieved because we need discontinuity to achieve the supremum. Next we can use the contrapositive of Weierstrass Theorem: if a continuous function doesn't achieve supremum on a set, then the set is not compact.

2. Continuous functions map compact sets to compact sets.
   Proof: $T : \mathsf{M} \to \mathsf{Y}$ continuous, $\mathsf{M} \subseteq \mathsf{X}$ is compact. Define $T(\mathsf{M}) := \{Tx : x \in \mathsf{M}\}$. We want to show that $T(\mathsf{M})$ is compact. Any sequence in $T(\mathsf{M})$ has the form $(Tx_n)_{n\geq 1}$ for some sequence $(x_n)_{n\geq 1}$ in M. By compactness of M, $(x_n)_{n\geq 1}$ has a convergent subsequence with limit $x_0 \in \mathsf{M}$. Observe that, along this subsequence, (Tx_n) converges to Tx_0 as T is continuous. Thus, $x_0 \in \mathsf{M} \implies Tx_0 \in T(\mathsf{M})$, so $T(\mathsf{M})$ is compact. ■