Problems to be handed in

1 A *semimetric* on a space X (which need not be a vector space) is a function $d : \mathsf{X} \times \mathsf{X} \to [0, \infty)$ with the following properties:

- $d(x, y) = d(y, x)$ for all $x, y \in \mathsf{X}$ (symmetry);
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in \mathsf{X}$ (triangle inequality);
- $d(x, x) = 0$ for every $x \in \mathsf{X}$ (weak zero property).

A *metric* on X is a semimetric d with the *strong zero property*, i.e., $d(x, y) = 0$ implies $x = y$. If d is a (semi)metric, then the pair (X, d) is referred to as a (semi)metric space.

(i) Let $(\mathsf{X}, \|\cdot\|)$ be a normed space. Prove that $d(x, y) := \|x - y\|$ is a metric.

(ii) Let X be a vector space. Give an example of a metric d on X that is not induced by a norm (i.e., there is no norm $\|\cdot\|$ on X that satisfies $\|x - y\| = d(x, y)$).

(iii) Let $(x_n)_{n\geq 1}$ be a sequence of elements of a metric space (X, d). We say that it converges to $x \in \mathsf{X}$ if

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Likewise, we say that $(x_n)_{n\geq 1}$ is a Cauchy sequence if, for any $\varepsilon > 0$, there exists some $N \geq 1$ such that

$$d(x_n, x_m) < \varepsilon, \qquad \forall m, n \geq N.$$

Prove that any convergent sequence is Cauchy. Give an example of a metric space in which the converse does not hold.

(iv) Let (X, d) be a metric space, and let $\mathsf{X}^\sharp$ be the space of all Cauchy sequences $\boldsymbol{x} = (x_n)_{n\geq 1}$ in X. Prove that, for any two $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{X}^\sharp$, the limit

$$d^\sharp(\boldsymbol{x}, \boldsymbol{y}) := \lim_{n\to\infty} d(x_n, y_n)$$

exists, and is a semimetric on $\mathsf{X}^\sharp$.

Note: This is the starting point of the *completion* procedure, where we take a metric space (X, d), embed it in $\mathsf{X}^\sharp$, and then pass to another space $\tilde{\mathsf{X}}$ consisting of equivalence classes of Cauchy sequences, where $\boldsymbol{x}$ and $\boldsymbol{y} \in \mathsf{X}^\sharp$ are declared to be equivalent if $d^\sharp(\boldsymbol{x}, \boldsymbol{y}) = 0$. It can then be shown that $\tilde{\mathsf{X}}$ can be equipped with a metric $\tilde{d}$, such that X embeds isometrically in $\tilde{\mathsf{X}}$ and $\tilde{\mathsf{X}}$ is complete, i.e., every Cauchy sequence of its elements is convergent.

2 Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and let $L^0(\mathbf{P})$ denote the space of all real-valued random variables X on Ω that are *almost surely finite*, i.e., those for which

$$\mathbf{P}[|X| < \infty] := \mathbf{P}\left[\bigcup_{n=1}^{\infty} \{|X| \leq n\}\right] = 1.$$

We can measure the "size" of a random variable $X \in L^0(\mathbf{P})$ using the L^0 *quasinorm*

$$|X|_{L^0} := \inf\left\{\varepsilon > 0 : \mathbf{P}[|X| > \varepsilon] \leq \varepsilon\right\}.$$

Note that $|X|_{L^0} \leq \lambda$ if and only if $\mathbf{P}[|X| > \lambda] \leq \lambda$. It can be shown, for instance, that $\mathbf{E}[|X|] < \infty$ implies that $X \in L^0(\mathbf{P})$, but the converse need not hold (e.g., consider the Cauchy random variable).

Prove the following:

(i) $d_0(X, Y) := |X - Y|_{L^0}$ is a semimetric on $L^0(\mathbf{P})$;

(ii) if $(X_n)_{n \geq 0}$ is a sequence of elements of $L^0(\mathbf{P})$ such that $|X_n - X|_{L^0} \xrightarrow{n \to \infty} 0$, then X_n converges to X *in probability*, i.e., for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbf{P}\left[|X_n - X| > \varepsilon\right] = 0.$$

3 A function $f : [0, 1] \to \mathbb{R}$ is *Lipschitz* if

$$\|f\|_{\mathsf{Lip}} := \sup\left\{\frac{|f(s) - f(t)|}{|s - t|} : s, t \in [0, 1],\ s \neq t\right\}$$

is finite. Evidently, any Lipschitz function is continuous. Prove that the set of all functions $f : [0, 1] \to \mathbb{R}$, such that

$$\|f\|_\infty + \|f\|_{\mathsf{Lip}} \leq 1,$$

is a compact subset of $(C[0, 1], \|\cdot\|_\infty)$.

4 Given two functions $f, g \in C[0, 1]$ satisfying $f(t) \leq g(t)$ for all $t \in [0, 1]$, the (f, g)-*bracket* is the set

$$[f, g] := \left\{x \in C[0, 1] : f(t) \leq x(t) \leq g(t),\ \forall t \in [0, 1]\right\}.$$

We say that $[f, g]$ is an ε-*bracket* if $\|f - g\|_\infty \leq \varepsilon$.

Let $\mathcal{F}$ be a subset of $(C[0, 1], \|\cdot\|_\infty)$ with the following property: For any $\varepsilon > 0$, there exist finitely many ε-brackets $[f_j, g_j]$, $1 \leq j \leq J < \infty$ (with f_j, g_j not necessarily in $\mathcal{F}$), such that

$$\mathcal{F} \subseteq \bigcup_{j=1}^{J} [f_j, g_j].$$

Prove that $\mathcal{F}$ is a relatively compact subset of $(C[0, 1], \|\cdot\|_\infty)$.