Problems to be handed in

1 A semimetric on a space $X$ (which need not be a vector space) is a function $d : X \times X \to [0, \infty)$ with the following properties:
   - $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry);
   - $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality);
   - $d(x, x) = 0$ for every $x \in X$ (weak zero property).

A metric on $X$ is a semimetric $d$ with the strong zero property, i.e., $d(x, y) = 0$ implies $x = y$. If $d$ is a (semi)metric, then the pair $(X, d)$ is referred to as a (semi)metric space.

(i) Let $(X, \| \cdot \|)$ be a normed space. Prove that $d(x, y) := \|x - y\|$ is a metric.

(ii) Let $X$ be a vector space. Give an example of a metric $d$ on $X$ that is not induced by a norm (i.e., there is no norm $\| \cdot \|$ on $X$ that satisfies $\|x - y\| = d(x, y)$).

(iii) Let $(x_n)_{n \geq 1}$ be a sequence of elements of a metric space $(X, d)$. We say that it converges to $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0.$$  

Likewise, we say that $(x_n)_{n \geq 1}$ is a Cauchy sequence if, for any $\varepsilon > 0$, there exists some $N \geq 1$ such that

$$d(x_n, x_m) < \varepsilon, \quad \forall m, n \geq N.$$  

Prove that any convergent sequence is Cauchy. Give an example of a metric space in which the converse does not hold.

(iv) Let $(X, d)$ be a metric space, and let $X^2$ be the space of all Cauchy sequences $x = (x_n)_{n \geq 1}$ in $X$. Prove that, for any two $x, y \in X^2$, the limit

$$d^\sharp(x, y) := \lim_{n \to \infty} d(x_n, y_n)$$  

exists, and is a semimetric on $X^2$.

*Note:* This is the starting point of the completion procedure, where we take a metric space $(X, d)$, embed it in $X^2$, and then pass to another space $\tilde{X}$ consisting of equivalence classes of Cauchy sequences, where $x$ and $y \in X^2$ are declared to be equivalent if $d^\sharp(x, y) = 0$. It can then be shown that $\tilde{X}$ can be equipped with a metric $\tilde{d}$, such that $X$ embeds isometrically in $\tilde{X}$ and $\tilde{X}$ is complete, i.e., every Cauchy sequence of its elements is convergent.
2 Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $L^0(P)$ denote the space of all real-valued random variables $X$ on $\Omega$ that are almost surely finite, i.e., those for which

$$P[|X| < \infty] := P\left(\bigcup_{n=1}^{\infty} \{ |X| \leq n \} \right) = 1.$$

We can measure the “size” of a random variable $X \in L^0(P)$ using the $L^0$ quasinorm

$$|X|_{L^0} := \inf \{ \varepsilon > 0 : P[|X| > \varepsilon] \leq \varepsilon \}.$$

Note that $|X|_{L^0} \leq \lambda$ if and only if $P[|X| > \lambda] \leq \lambda$. It can be shown, for instance, that $E[|X|] < \infty$ implies that $X \in L^0(P)$, but the converse need not hold (e.g., consider the Cauchy random variable).

Prove the following:

(i) $d_0(X, Y) := |X - Y|_{L^0}$ is a semimetric on $L^0(P)$;

(ii) if $(X_n)_{n \geq 0}$ is a sequence of elements of $L^0(P)$ such that $|X_n - X|_{L^0} \xrightarrow{n \to \infty} 0$, then $X_n$ converges to $X$ in probability, i.e., for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P[|X_n - X| > \varepsilon] = 0.$$

3 A function $f : [0, 1] \to \mathbb{R}$ is Lipschitz if

$$\|f\|_{\text{Lip}} := \sup \left\{ \frac{|f(s) - f(t)|}{|s - t|} : s, t \in [0, 1], s \neq t \right\}$$

is finite. Evidently, any Lipschitz function is continuous. Prove that the set of all functions $f : [0, 1] \to \mathbb{R}$, such that

$$\|f\|_{\infty} + \|f\|_{\text{Lip}} \leq 1,$$

is a compact subset of $(C[0, 1], \| \cdot \|_{\infty})$.

4 Given two functions $f, g \in C[0, 1]$ satisfying $f(t) \leq g(t)$ for all $t \in [0, 1]$, the $(f, g)$-bracket is the set

$$[f, g] := \{ x \in C[0, 1] : f(t) \leq x(t) \leq g(t), \forall t \in [0, 1] \}.$$

We say that $[f, g]$ is an $\varepsilon$-bracket if $\|f - g\|_{\infty} \leq \varepsilon$.

Let $\mathcal{F}$ be a subset of $(C[0, 1], \| \cdot \|_{\infty})$ with the following property: For any $\varepsilon > 0$, there exist finitely many $\varepsilon$-brackets $[f_j, g_j]$, $1 \leq j \leq J < \infty$ (with $f_j, g_j$ not necessarily in $\mathcal{F}$), such that

$$\mathcal{F} \subseteq \bigcup_{j=1}^{J} [f_j, g_j].$$

Prove that $\mathcal{F}$ is a relatively compact subset of $(C[0, 1], \| \cdot \|_{\infty})$. 
