# The Linear Quadratic Regulator (LQR) Problem: Finite and Infinite Time

1) Review $\dot{x} = f(t, x, u)$ $\quad x(t) \in \mathbb{R}^n;\ u(t) \in \mathbb{R}^m$

$$J(t_1, t_0, x, u(\cdot)) := \int_{t_0}^{t_1} q(t, x(t), u(t))\,dt + p(x(t_1))$$

to be minimized over all "admissible" controls $u(\cdot)$ subject to

$$\dot{x}(t) = f(t, x(t), u(t)), \quad t \in [t_0, t_1], \quad x(t_0) = x$$

Bellman fcn: $V(t, x) := \min\limits_{u(\cdot)} J(t_1, t, x, u(\cdot))$

$$t_0 \le t \le t_1$$

Under regularity conditions [e.g., $V$ is $C^1$], the Hamilton-Jacobi-Bellman eqn.

$$\frac{\partial}{\partial t} V(t, x) = -\min_{u \in \mathbb{R}^m} \left\{ q(t, x, u) + \frac{\partial}{\partial x} V(t, x) f(t, x, u) \right\}$$

for $t \in (t_0, t_1]$, $x \in \mathbb{R}^n$, $\quad V(t_1, x) = p(x)$

gives an optimal control $k(t, x)$ that minimizes

$$q(t, x, u) + \frac{\partial V}{\partial x}(t, x) f(t, x, u)$$

for fixed $(t, x) \in (t_0, t_1] \times \mathbb{R}^n$.

If $V(t, x)$ is a $C^1$ solution of HJB eqn, then it is the Bellman fcn, and the optimal control is given by $\bar{u}(t) = k(t, x(t))$ along the optimal trajectory.

Uniqueness: if $k(t, x)$ is such that

$$q(t, x, k(t, x)) + \frac{\partial}{\partial x} V(t, x) f(t, x, k(t, x))$$

$$< q(t, x, u) + \frac{\partial}{\partial x} V(t, x) f(t, x, u) \quad \forall u \in \mathbb{R}^m$$

$$\text{s.t. } u \ne k(t, x)$$

then $J(t_1, t_0, x, u(\cdot)) > V(t_0, x)$

$$\equiv \min_{u(\cdot)} J(t_1, t_0, x, u(\cdot))$$

for all $u(\cdot)$ that are not equal to $k(t,x)$ almost everywhere.

## 2) Finite-time LQR problem

LTV system: $\dot{x}(t) = A(t)x(t) + B(t)u(t)$

$t_0 \leq t \leq t_1, \qquad x(t_0) = x$

$$J(t_1, t_0, x, u(\cdot)) := \int_{t_0}^{t_1} \{u(t)^T R(t) u(t) + x(t)^T Q(t) x(t)\} dt + x(t_1)^T S x(t_1)$$

where $R(t) = R(t)^T > 0$ (in $\mathbb{R}^{m \times m}$)

$Q(t) = Q(t)^T \geq 0$ (in $\mathbb{R}^{n \times n}$)

$S = S^T \geq 0$ (in $\mathbb{R}^{n \times n}$)

Goal: determine optimal control law, compute $V(t, x)$.

Analysis: $V(t, x) = x^T P(t) x$ (ansatz)

$t_0 \leq t \leq t_1, \qquad V(t_1, x) = x^T S x \equiv p(x)$

$\Rightarrow P(t_1) = S$, and

$$x^T P(t_0) x \equiv \min_{u(\cdot)} J(t_1, t_0, x, u(\cdot))$$

$$\frac{\partial}{\partial t} V(t, x) = -\min_{u \in \mathbb{R}^m} \{u^T R(t) u + x^T Q(t) x + \frac{\partial}{\partial x} V(t, x) (A(t)x + B(t)u)\}$$

$\frac{\partial}{\partial t} V(t, x) = x^T \dot{P}(t) x$

$\frac{\partial}{\partial x} V(t, x) = 2x^T P(t)$

assume: $[P(t) = P(t)^T]$

For any $x \in \mathbb{R}^n$:

$$x^T \dot{P} x = -\min_{u \in \mathbb{R}^m} \{u^T R u + x^T Q x + 2x^T P (Ax + Bu)\}$$

$$F(u) := x^T Q x + x^T P A x + x^T A^T P x$$
$$+ u^T R u + 2x^T P B u$$

$$\min_{u \in \mathbb{R}^m} F(u) = x^T Q x + x^T (PA + A^T P) x$$
$$+ \min_{u \in \mathbb{R}^m} \{u^T R u + 2u^T B^T P x\}$$

Here, $R(t) = R(t)^T > 0$, so $u \mapsto u^T R u + 2u^T B^T P x$ is strongly convex, so the minimizing $u(t,x)$ is unique, and solves

$$2R(t) u(t,x) = -2B(t)^T P(t) x$$
$$u(t,x) = -R(t)^{-1} B(t)^T P(t) x$$
$$=: F(t) x .$$

$$F(u(t,x)) = x^T F^T R F x + 2x^T F^T B^T P x$$

$$= x^T P B R^{-1} R R^{-1} B^T P x - 2x^T P B R^{-1} B^T P x$$
$$= -x^T P B R^{-1} B^T P x$$

$$\Rightarrow x^T \dot{P} x = -x^T (PA + A^T P + Q - PBR^{-1} B^T P) x$$
$$t \in (t_0, t_1], \quad P(t_1) = S$$

for all $x \in \mathbb{R}^n$, so $P(t)$ has to solve

$$\boxed{\dot{P} = +PBR^{-1}B^T P - PA - A^T P - Q, \quad P(t_1) = S}$$

- so, $P(t)$ is a solution of this Riccati DE.

Observations:

1) The RDE has a solution $P(t) \in \mathbb{R}^{n \times n}$, which is unique.

2) If $P(t)$ solves the RDE, then $P(t)^T$ also solves it, so $P(t) = P(t)^T$ by uniqueness

3) $P(t) \geq 0$ : $V(t,x) = x^T P(t) x$ is the Bellman fcn for the LQR problem, so

$$V(t,x) = x^T P(t) x$$
$$= \min_{u(\cdot)} \left\{ \int_t^{t_1} \{ u(s)^T R(s) u(s) + x(s)^T Q(s) x(s) \} ds + x(t_1)^T S x(t_1) \right\}$$
$$\geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

Therefore, $k(t,x) = F(t)x = -R(t)^{-1} B(t)^T P(t) x$ is the unique optimal control, and

$V(t_0, x) = x^T P(t_0) x$ is the minimal cost.

Closed-loop system (at optimality):

$$\dot{x}(t) = (A(t) + B(t) F(t)) x(t)$$
$$x(t_0) = x$$

3) Infinite-time (steady-state) LQR problem

LTI: $\dot{x} = Ax + Bu$

$x(0) = x, \quad t \geq 0$

Cost: $J_\infty(x, u(\cdot)) := \int_0^\infty \{ u(t)^T R u(t) + x(t)^T Q x(t) \} dt$

to be minimized over admissible controls $u(\cdot)$, where $R = R^T > 0$, $Q = Q^T \geq 0$.

Bellman fcn: $V(x) := \min_{u(\cdot)} J_\infty(x, u(\cdot))$.

Assumption: $(A, B)$ is a controllable pair.
(this guarantees that, for any $x \in \mathbb{R}^n$, $\exists\, u(\cdot)$ s.t.
$$J_\infty(x, u(\cdot)) < \infty).$$

Analysis:
Fix some $\tau > 0$, consider
$$\min_{u(\cdot)} J(\tau, 0, x, u(\cdot))$$
with $q(t, x, u) = u^T R u + x^T Q x$, $\quad p(x) = 0$

RDE: $\quad P_\tau(\cdot) = P_\tau(\cdot)^T \geq 0$
$$\dot{P}_\tau = P_\tau B R^{-1} B^T P_\tau - P_\tau A - A^T P_\tau - Q$$
$$0 \leq t \leq \tau \qquad P_\tau(\tau) = 0$$

Let $\Pi(t) := P_\tau(\tau - t)$ for $0 \leq t \leq \tau$
So, $\Pi(0) = P_\tau(\tau) = 0$, and
$$\dot{\Pi} = -\Pi B R^{-1} B^T \Pi + \Pi A + A^T \Pi + Q$$
$$0 \leq t \leq \tau, \qquad \Pi(0) = 0$$

If we do this for every $\tau > 0$, then we see that we can define $\Pi(t) = \Pi(t)^T \geq 0 \quad \forall t \geq 0$, s.t.
$$\dot{\Pi} = -\Pi B R^{-1} B \Pi + \Pi A + A^T \Pi + Q, \quad t \geq 0$$
$$\Pi(0) = 0.$$

We will show that $\Pi := \lim_{t \to \infty} \Pi(t)$ exists.

$x^T \Pi(\tau) x = V_\tau(0, x)$, where $V_\tau$ is the Bellman fcn for the LQR problem on $[0, \tau]$, with

$q(x, u) = u^T R u + x^T Q x$, $\quad p(x) = 0$

Fix $\sigma > \tau \geq 0$.

Claim: $\quad x^T \Pi(\sigma) x \geq x^T \Pi(\tau) x \geq 0$

$$
\begin{aligned}
x^T \Pi(\sigma) x &= V_\sigma(0, x) \\
&= J(\sigma, 0, x, u_\sigma(\cdot)) \qquad u_\sigma(\cdot) - \text{optimal on } [0, \sigma] \\
&= J(\tau, 0, x, u_\sigma(\cdot)) \\
&\quad + \int_\tau^\sigma q(x(t), u_\sigma(t)) \, dt \\
&\geq J(\tau, 0, x, u_\sigma(\cdot)) \\
&\geq V_\tau(0, x) \\
&= x^T \Pi(\tau) x \qquad \forall x \in \mathbb{R}^n
\end{aligned}
$$

$\Rightarrow$ for each $x \in \mathbb{R}^n$, $t \mapsto x^T \Pi(t) x$ is bounded below by 0 and nondecreasing.

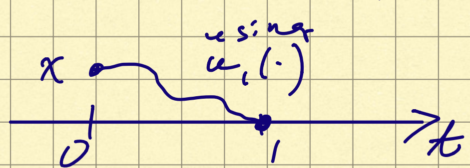
Claim: $(A, B)$ controllable pair $\Rightarrow$

$$\Pi = \lim_{t \to \infty} \Pi(t)$$

exists, and solves Algebraic Riccati Eqn

$$\Pi B R^{-1} B^T \Pi - \Pi A - A^T \Pi - Q = 0$$

Fix $x \in \mathbb{R}^n$. By controllability $\exists u_1(\cdot)$ s.t.




Take $u(t) := \begin{cases} u_1(t), & 0 \leq t \leq 1 \\ 0, & t > 0 \end{cases}$

Then $J_\infty(x, u(\cdot))$

$$= \int_0^1 \{u_1(t)^T R u_1(t) + x(t)^T Q x(t)\} dt < \infty$$

$\Rightarrow \quad \min_{u(\cdot)} J_\infty(x, u(\cdot)) < \infty$

Then, $\forall \tau > 0$,

$$x^T \Pi(\tau) x = V_\tau(0, x)$$
$$\leq J(\tau, 0, x, u(\cdot))$$
$$\leq J_\infty(0, x, u(\cdot)) < \infty$$

$\Rightarrow \quad 0 \leq x^T \Pi(\tau) \leq x^T \Pi(\sigma) x \leq J_\infty(x, u(\cdot)) < \infty$

for all $0 \leq \tau < \sigma$

So, $\lim_{t\to\infty} x^T \Pi(t) x$ exists for each $x$.

Thus, $\Pi := \lim_{t\to\infty} \Pi(t)$ exists

[apply to $x = e_i$, $x = e_i + e_j$ $\forall i, j \in \{1, \dots, n\}$, where $e_1, \dots, e_n \in \mathbb{R}^n$ are the standard basis vectors in $\mathbb{R}^n$]

$$\dot{\Pi}(t) = -\Pi(t) B R^{-1} B^T \Pi(t) + \Pi(t) A + A^T \Pi(t) + Q$$
$$\Pi(0) = 0$$

$\lim_{t\to\infty} \dot{\Pi}(t)$ exists, $= 0$ :

so $\boxed{\Pi B R^{-1} B^T \Pi - \Pi A - A^T \Pi - Q = 0}$

**Claim**: $k(x) = -Fx$, where $F := R^{-1} B^T \Pi$,

is the unique optimal control that achieves $V(x)$ (Bellman fcn) at each $x$.

**Proof**: next lecture.