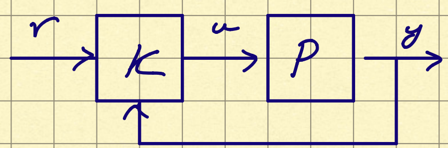
Model Reference Adaptive Control (MRAC)

Goal: to (approximately) reproduce some desired i/o behavior w/o knowledge of the plant (up to membership in some class).



$r \to y$ specified

need to design K to implement it

when P is unknown, two approaches exist:

- **direct**: tune the parameters of K directly
- **indirect**: estimate params of P online, use this to tune K

Example 1st-order linear plant, stable ref. model

plant: $\dot{y} = ay + bu$ ref: $\dot{y}_m = -a_m y_m + b_m r$

$a \in \mathbb{R},\ b > 0$ $a_m > 0,\ b_m \neq 0;\ r \in L_\infty$

controller: $u = -ky + lr$

Known plant: $\dot{y} = ay + b(-ky + lr)$

$= (a - bk)y + blr$

controller parametrization: $\left.\begin{aligned} a - bk &= -a_m \\ bl &= b_m \end{aligned}\right\}\ k = \frac{a_m + a}{b},\ l = \frac{b_m}{b}$

Adaptive controller: $u = -\hat{k} y + \hat{l} r$

where $\dot{\hat{k}}, \dot{\hat{l}}$ are to be designed

Basic goal: $y(t) - y_m(t) \to 0$ as $t \to \infty$

for any given $r \in L_\infty$

Direct MRAC

- plant reparametrization

$\dot{y} = ay + bu$

$u = -\hat{k}y + \hat{l}r$ $\longrightarrow$ $\dot{y} = -a_m y + b_m r + b(-\hat{k}y + \hat{l}r + ky - lr)$

$\dot{\hat{k}} = ?$ $\qquad k = \frac{a_m + a}{b}, \quad l = \frac{b_m}{b}$

$\dot{\hat{l}} = ?$

Define the estimation errors $\tilde{k} := \hat{k} - k, \quad \tilde{l} := \hat{l} - l$

$\dot{y} = -a_m y + b_m r + b(-\tilde{k}y + \tilde{l}r)$

Tracking error: $e := y_m - y$

- analysis: Lyapunov methods

$V(e, \tilde{k}, \tilde{l}) := \frac{1}{2}\left(\frac{e^2}{b} + \frac{1}{\gamma}(\tilde{k}^2 + \tilde{l}^2)\right),$

where $\gamma > 0$ is a tunable constant

- candidate LF

$\dot{V} = \frac{e\dot{e}}{b} + \frac{\tilde{k}\dot{\tilde{k}}}{\gamma} + \frac{\tilde{l}\dot{\tilde{l}}}{\gamma}$ $\qquad \dot{\tilde{k}} = \dot{\hat{k}}$, $\dot{\tilde{l}} = \dot{\hat{l}}$ to be designed

$= \frac{e\dot{e}}{b} + \frac{\tilde{k}\dot{\hat{k}}}{\gamma} + \frac{\tilde{l}\dot{\hat{l}}}{\gamma}$

$\dot{e} = \dot{y}_m - \dot{y}$

$= -a_m y_m + b_m r - (-a_m y + b_m r + b(-\tilde{k}y + \tilde{l}r))$

$= -a_m(y_m - y) - b(-\tilde{k}y + \tilde{l}r)$

$= -a_m e - b(-\tilde{k}y + \tilde{l}r)$

$\dot{V} = \frac{e}{b}\left(-a_m e - b(-\tilde{k}y + \tilde{l}r)\right) + \frac{1}{\gamma}\tilde{k}\dot{\hat{k}} + \frac{1}{\gamma}\tilde{l}\dot{\hat{l}}$

$= -\frac{a_m}{b}e^2 + \tilde{k}ey - \tilde{l}er + \frac{1}{\gamma}\tilde{k}\dot{\hat{k}} + \frac{1}{\gamma}\tilde{l}\dot{\hat{l}}$

$$\dot V = -\frac{a_m}{b} e^2 + \tilde k\big(\underbrace{ey + \tfrac{1}{\gamma}\dot{\hat k}}_{\text{want}=0}\big) + \tilde \ell\big(\underbrace{-er + \tfrac{1}{\gamma}\dot{\hat \ell}}_{\text{want}=0}\big)$$

$$\Longrightarrow \quad \dot{\hat k} = -\gamma e y \qquad\qquad \boxed{\dot V = -\frac{a_m}{b} e^2 \le 0}$$
$$\dot{\hat \ell} = \gamma e r$$

$$0 \le V(e(t), \tilde k(t), \tilde \ell(t)) \le V(e(0), \tilde k(0), \tilde \ell(0))$$

$\Rightarrow \quad e, \tilde k, \tilde \ell \in L_\infty \quad$ so $\quad \hat k, \hat \ell \in L_\infty$

$$\int_0^t e^2(s)\,ds \le \frac{b}{a_m} V(0) \qquad \forall t \ge 0$$
$$\Rightarrow \quad e \in L_2$$

How do we guarantee $e \to 0$? Need $\dot e \in L_\infty$

$$\dot e = -a_m e + b(-\tilde k y + \tilde \ell r)$$

$e \in L_\infty$, $\tilde k, \tilde \ell \in L_\infty$, $r \in L_\infty$

$y = y_m - e \in L_\infty$ (because ref. model is stable, $r \in L_\infty \Rightarrow y_m \in L_\infty$)

So, $y_m(t) - y(t) \to 0$ as $t \to \infty$ for any bdd ref. input r, by Barbalat.

What about $\tilde k, \tilde \ell \to 0$?

Need a PE condition on r.

$$\dot e = -a_m e + b \tilde k y - b \tilde \ell r$$
$$\dot{\tilde k} = \gamma e y$$
$$\dot{\tilde \ell} = -\gamma e r$$

state $x = \begin{pmatrix} e \\ \tilde k \\ \tilde \ell \end{pmatrix}$, output $y = e$

$$\begin{pmatrix} \dot{e} \\ \dot{\tilde{k}} \\ \dot{\tilde{l}} \end{pmatrix} = \begin{pmatrix} -a_m & by & -br \\ \gamma y & 0 & 0 \\ -\gamma r & 0 & 0 \end{pmatrix} \begin{pmatrix} e \\ \tilde{k} \\ \tilde{l} \end{pmatrix} \qquad \dot{x} = A(t)x$$

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e \\ \tilde{k} \\ \tilde{l} \end{pmatrix}$$

Take $r(t) = \sin(\omega t)$

$y(t) = A \sin(\omega t + \alpha)$

$$A(t) = \begin{pmatrix} -a_m & bA\sin(\omega t+\alpha) & -b\sin(\omega t) \\ \gamma A\sin(\omega t+\alpha) & 0 & 0 \\ -\gamma\sin(\omega t) & 0 & 0 \end{pmatrix}$$

Want to show that $(A(t), c)$, $c = (1, 0, 0)$ is UCO, so then $e, \tilde{k}, \tilde{l} \to 0$

Add output injection: $(A(t), c) \longrightarrow (A(t) + L(t)c, c)$, show this new system is UCO.

A good choice of $L(t)$ is to give

$$A(t) + L(t)c = \begin{pmatrix} -a_m & bA\sin(\omega t+\alpha) & -b\sin(\omega t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

here the new state $\dot{x} = (A(t) + L(t)c)x$

$y = cx \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$x_2 = \text{const}, \qquad x_3 = \text{const}$

$$y(t) = x_1(t) = e^{-a_m t} x_1(0) + v(t) \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

$\geq \alpha \, |x(t)|^2$ for some $\alpha > 0$

if $r(t) = \sin(\omega t)$.

Bottom line: $e \to 0$ for bdd r

$\hat{k} \to k$, $\hat{l} \to l$ for sinusoidal r

Indirect MRAC

Keep $u = -\hat{k}y + \hat{l}r$,

but now update estimates $\hat{a}$, $\hat{b}$, take $\hat{k}, \hat{l}$ as functions of these.

$$k = \frac{a_m + a}{b}, \qquad l = \frac{b_m}{b}$$

$$\hat{k} = \frac{a_m + \hat{a}}{\hat{b}}, \qquad \hat{l} = \frac{b_m}{\hat{b}}$$

(beware: $\hat{b}$ close to 0 may lead to loss of stabilizability)

$$\dot{y} = ay + b(-\hat{k}y + \hat{l}r)$$

$$\dot{y}_m = -a_m y_m + b_m r \quad \Leftarrow \text{ implementable}$$

Here, can use y_m as an "estimate" of y:

$$\underline{\dot{\hat{y}} = -a_m(\hat{y} - y) + \hat{a}y + \hat{b}u}$$

– we can use y_m as $\hat{y}$

$$\begin{aligned}
\dot{y}_m &= -a_m y_m + b_m r \\
&= -a_m(y_m - y) - a_m y + b_m r \\
&= -a_m(y_m - y) - (a_m + \hat{a})y + \hat{a}y + b_m r \\
&= -a_m(y_m - y) + \hat{a}y - \hat{b}\hat{k}y + \hat{b}\hat{l}r \\
&= -a_m(y_m - y) + \hat{a}y + \hat{b}\underbrace{(-\hat{k}y + \hat{l}r)}_{=u}
\end{aligned}$$

$$\Rightarrow \quad \underline{\dot{y}_m = -a_m(y_m - y) + \hat{a}y + \hat{b}u}$$

Tracking error: $e := y_m - y$

Candidate LF: $V(e, \tilde{a}, \tilde{b}) := \frac{1}{2}\left(e^2 + \frac{1}{\gamma}(\tilde{a}^2 + \tilde{b}^2)\right)$

$\tilde{a} := \hat{a} - a, \quad \tilde{b} := \hat{b} - b$

$\dot{V} = e\dot{e} + \frac{1}{\gamma}\tilde{a}\dot{\hat{a}} + \frac{1}{\gamma}\tilde{b}\dot{\hat{b}}$

$\dot{e} = \dot{y}_m - \dot{y}$

$= -a_m e + \hat{a}y + \hat{b}u - (ay + bu)$

$= -a_m e + \tilde{a}y + \tilde{b}u$

$\dot{V} = -a_m e^2 + \tilde{a}ey + \tilde{b}eu + \frac{1}{\gamma}\tilde{a}\dot{\hat{a}} + \frac{1}{\gamma}\tilde{b}\dot{\hat{b}}$

$= -a_m e^2 + \tilde{a}(ey + \frac{1}{\gamma}\dot{\hat{a}}) + \tilde{b}(eu + \frac{1}{\gamma}\dot{\hat{b}})$

Take $\dot{\hat{a}} = -\gamma ey, \quad \dot{\hat{b}} = -\gamma eu$

$\dot{V} = -a_m e^2 \leq 0$

$e, \tilde{a}, \tilde{b} \in L_\infty \implies e, \hat{a}, \hat{b} \in L_\infty$

$e \in L_2$

We need to show $e \to 0$ as $t \to \infty$.

$\dot{e} = -a_m e + \tilde{a}y + \tilde{b}u$, need $\dot{e}$ bdd to apply Barbalat

$e \in L_\infty$, $\tilde{a} \in L_\infty$, $y = y_m - e \in L_\infty$ (since $e \in L_\infty$, $y_m \in L_\infty$)

$\tilde{b} \in L_\infty$

Can we say $u \in L_\infty$? $u := -\hat{k}y + \hat{l}r$

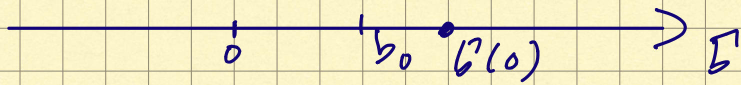
$\hat{k} = \frac{a_m + \hat{a}}{\hat{b}}, \quad \hat{l} = \frac{b_m}{\hat{b}}$

Issue: u can become unbounded if $\hat b \sim 0$.

There are multiple ways of fixing this; one involves projections.

Suppose we know $b \geq b_0 > 0$ (b_0 known). We can modify the dynamics of $\hat b$ to force it into the constraint set $\{\hat b \geq b_0\}$:

$$\dot{\hat b} = \begin{cases} -\gamma e u, & \hat b > b_0 \text{ or } (\hat b = b_0 \text{ and } eu < 0) \\ 0, & \text{otherwise} \end{cases}$$



$$\dot V = -a_m e^2 + \underbrace{\frac{1}{\gamma} \tilde b \dot{\hat b}}_{\leq 0} \quad \text{with the modified dynamics}$$