Sufficiently Rich Inputs and Parameter Identification

Recall: $z = \theta^T \phi$ (linear parametrization)

θ : system parameters (freq. domain description)

$z = y^{(n)}$, $\phi = (u^{(n-1)}, \dots, u', u, -y^{(n-1)}, \dots, -y', -y)$

(filtered versions, n: model order)

Observe $(u(t), y(t))$, $t \geq 0$

$\hat{\theta}(t)$: estimates of θ based on $(z(s), \phi(s))$, $0 \leq s \leq t$
(e.g. use gradient flow or online LS)

Output prediction error: $\hat{z}(t) - z(t)$
$= (\hat{\theta}(t) - \theta)^T \phi(t)$

Param. identification error: $\hat{\theta}(t) - \theta$

$\hat{\theta}(t) \to \theta$ under a Persistency of Excitation assumption on ϕ:

$$\int_{t_0}^{t_0+T_0} \phi(t)\phi(t)^T \, dt \geq \alpha_0 T_0 I \qquad \forall t_0 \geq 0$$

for some $\alpha_0, T_0 > 0$

Note: ϕ consists of filtered versions of u (input) and y (the output) but the only signal we have direct control over is u.

Need conditions on u alone to guarantee PE of ϕ.

$$\boxed{u \text{ sufficiently rich}} \longrightarrow \boxed{\phi \text{ is PE for some } \alpha_0, T_0} \xrightarrow{\text{already covered}} \boxed{\hat{\theta} \to \theta}$$

Example ① 1st order plant

$$\dot{y} = -ay + bu$$

$a > 0$ (stable system)
$b \in \mathbb{R}$ (arbitrary)

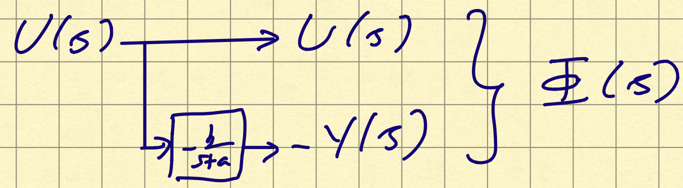
$u \in L_\infty$ (bdd input)

What makes u "sufficiently rich"?

Ignore filtering: $z = \dot{y}$ $\phi = \begin{pmatrix} u \\ -y \end{pmatrix}$, $\theta = \begin{pmatrix} b \\ a \end{pmatrix}$

$$H(s) = \frac{b}{s+a}$$

$$\phi = \begin{pmatrix} 1 \\ -\frac{b}{s+a} \end{pmatrix} u$$



PE: $$\int_{t_0}^{t_0+T_0} \phi(t)\phi(t)^T dt = \int_{t_0}^{t_0+T} \begin{pmatrix} u(t) \\ -y(t) \end{pmatrix} \begin{pmatrix} u(t) & -y(t) \end{pmatrix} dt$$

What is a good choice of $u(t)$?

- try a constant input, $u(t) = c$, $t \geq 0$

$$\dot{y} = -ay + bc$$
$$= -a\left(y - \frac{bc}{a}\right)$$

Let $\tilde{y}(t) := y(t) - \frac{bc}{a}$

$$\dot{\tilde{y}} = \dot{y} = -a\left(y - \frac{bc}{a}\right) = -a\tilde{y} \iff \tilde{y}(t) = \tilde{y}(0)e^{-at}$$

$$y(t) = \underbrace{\frac{bc}{a}}_{\substack{\text{depends} \\ \text{only on} \\ b/a}} + \underbrace{e^{-at}\left(y(0) - \frac{bc}{a}\right)}_{\text{decays exp. fast}}$$

— can't identify both a and b if we inject constant input.

More directly:

$$\phi(t)\phi(t)^T = \begin{pmatrix} u(t) \\ -y(t) \end{pmatrix} \begin{pmatrix} u(t) & -y(t) \end{pmatrix}$$

$u(t) = c$ (const)

$y(t) = \frac{bc}{a} + \tilde{y}(t)$ $\qquad$ $\tilde{y}(t)$ decays exponentially fast

$$\phi\phi^T = \begin{pmatrix} c \\ -\frac{bc}{a} - \tilde{y} \end{pmatrix} \begin{pmatrix} c & -\frac{bc}{a} - \tilde{y} \end{pmatrix}$$

$$= \begin{pmatrix} c^2 & -\frac{bc^2}{a} - c\tilde{y} \\ -\frac{bc^2}{a} - c\tilde{y} & \left(\frac{bc}{a} + \tilde{y}\right)^2 \end{pmatrix}$$

$$= \begin{pmatrix} c^2 & -\frac{bc^2}{a} - c\tilde{y} \\ -\frac{bc^2}{a} - c\tilde{y} & \frac{b^2c^2}{a^2} + \frac{2bc}{a}\tilde{y} + \tilde{y}^2 \end{pmatrix}$$

$\tilde{y}(t) = e^{-at}\tilde{y}(0)$

$$= \underbrace{\begin{pmatrix} c^2 & -\frac{bc^2}{a} \\ -\frac{bc^2}{a} & \frac{b^2c^2}{a^2} \end{pmatrix}}_{\det \equiv 0} + \underbrace{\begin{pmatrix} 0 & -c\tilde{y}(t) \\ -c\tilde{y}(t) & \frac{2bc}{a}\tilde{y}(t) + \tilde{y}^2(t) \end{pmatrix}}_{\det = -c^2\tilde{y}^2(t)}$$

$\to 0$ exp. fast

Try some other inputs (keep them bdd)

Sinusoids!

$u(t) = \sin \omega t$ $\qquad$ $\omega \neq 0$ - constant freq.

$$y(t) = e^{-at}y(0) + \int_0^t e^{-a(t-s)} \sin(\omega s)\, ds$$

$$\simeq A \sin(\omega t + \alpha) \qquad \text{(in steady state)}$$

where A and α are constants that depend on a, b

$$\phi(t) = \begin{pmatrix} \sin \omega t \\ A \sin(\omega t + \alpha) \end{pmatrix}$$ is PE (homework)

Ex. 2: 2nd-order plant

$$Y(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} U(s)$$

($a_1, a_0 > 0$: stable, by Routh-Hurwitz)

$$\theta = (b_1, b_0, a_1, a_0)^T \in \mathbb{R}^4$$

$$u \longrightarrow \boxed{G(s)} \longrightarrow y$$

$u(t) = \sin(\omega t) \quad \rightarrow \quad y(t) = A \sin(\omega t + \alpha)$

Linearity: $u(t) = \sin(\omega_1 t) + \sin(\omega_2 t) \qquad \omega_1 \neq \omega_2$

$$y(t) = A_1 \sin(\omega_1 t + \alpha_1) + A_2 \sin(\omega_2 t + \alpha_2)$$

Here: $z(t) = \ddot{y}(t)$, $\phi(t) = (u'(t), u(t), -y'(t), -y(t))^T$

General idea: for an nth-order SISO LTI system, an input u is sufficiently rich if it contains at least n distinct frequencies (assuming no pole-zero cancellations)

$$\int_0^T \phi(t)\phi(t)^T dt$$ – "autocovariance" of ϕ

– examine in Fourier domain (spectral measures)

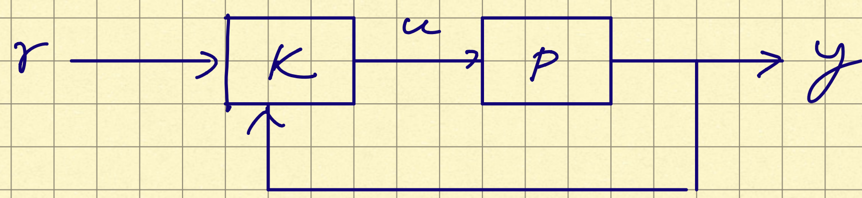
Caveats:

- parameter identification ($\hat{\theta} \rightarrow \theta$) is <u>not</u> necessary for control: PE may conflict w/ control goals
- parameter 'estimation' may still be useful even w/o PE
- PE may be of use in settings like reference matching

Model Reference Adaptive Control (MRAC)

Goal: get the closed-loop system (w/ unknown plant) to reproduce i/o behavior according to a given reference model.

Recall:



Ex.:

plant $\dot{y} = ay + bu$

$a \in \mathbb{R}$ (unknown, no assumption of stability)

$b > 0$ (sign of b known)

reference: $\dot{y}_m = -a_m y_m + b_m r$

$a_m > 0$ (stable reference)

$r \in L_\infty$

$\left.\right\}$ a_m, b_m : given desired closed-loop system specs

control: $u = -\underbrace{k}_{\text{feedback gain}} y + \underbrace{l}_{\text{feedforward gain}} r$

If a, b are known, k and l can be specified:

$$\dot{y} = ay + bu \qquad u = -ky + lr$$

$$= (a - bk)y + blr$$

Want: $a - bk = -a_m$, $bl = b_m$ $\longleftrightarrow$ $k = \frac{a + a_m}{b}$, $l = \frac{b_m}{b}$ $(b > 0)$

Since a, b are unknown, let's use:

$$u = -\hat{k}y + \hat{l}r \qquad \hat{k}, \hat{l} \text{ – dynamic estimates}$$

MRAC: design update laws $\dot{\hat{k}}, \dot{\hat{l}}$

- <u>direct</u> : update $\hat{k}, \hat{l}$ (control params) directly
- <u>indirect</u>: estimate a, b via $\hat{a}, \hat{b}$, use these to generate $\hat{k}, \hat{l}$

Direct MRAC

- reparametrize

$$\dot{y} = ay + bu$$
$$= -a_m y + b_m r + (a + a_m) y + bu - b_m r$$
$$= -a_m y + b_m r + bky + bu - blr$$

$$\dot{y} = -a_m y + b_m r + b\underbrace{(u + ky - lr)}_{= -\hat{k}y + \hat{l}r}$$

$$\boxed{\dot{y} = -a_m y + b_m r + b(-\tilde{k}y + \tilde{l}r)}$$

$\tilde{k} := \hat{k} - k$
$\tilde{l} := \hat{l} - l$

$e := y_m - y$

<u>Preview:</u> candidate Lyapunov fcn

$$V(e, \tilde{k}, \tilde{l}) := \frac{1}{2}\left(\frac{e^2}{b} + \frac{\tilde{k}^2}{\gamma} + \frac{\tilde{l}^2}{\gamma}\right),$$

where $\gamma > 0$ is a tunable constant