Backstepping

- a general technique for constructing CLF's and cont. stabilizing controllers for certain types of nonlinear systems
- 1978: Meilakhs (in a nonadaptive context)
  Feuer and Morse (adaptive control, MRAC)
- 1990s: systematic use + term "backstepping"
  Krstic, Kanellakopoulos, Kokotovic

Recall: $\dot{x} = f(x,u)$ $\qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$

$V(x)$ is a CLF if

$$\inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x} f(x,u) \right\} < 0 \quad \text{for all } x \neq 0$$

[assumption: $f(0,0) = 0$]

Control-affine systems: $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$

where $f: \mathbb{R}^n \to \mathbb{R}^n$
$g_i: \mathbb{R}^n \to \mathbb{R}^n \qquad i = 1, \dots, m$

CLF: for $x \neq 0$,

$$\left(\frac{\partial V}{\partial x} g_1(x), \dots, \frac{\partial V}{\partial x} g_m(x) \right) = 0 \Rightarrow \frac{\partial V}{\partial x} f(x) < 0$$

Then $\exists$ a cont. stabilizing controller $u = k(x)$.

Ex.: $\dot{x} = -x^3 + u \qquad x, u \in \mathbb{R}$

$V(x) = \frac{x^2}{2}$ is a CLF

Many stabilizing controllers are available:

- $u = x^3 - x$ (feedback linearization)
- $u = -x$ $\qquad \dot{V} = x\dot{x} = x(-x^3 - x) = -x^4 - x^2 < 0$
- $u \equiv 0$ $\qquad \dot{V} = -x^4 < 0$

- $u = -x^3 - x\sqrt{1+x^2}$ (universal formula)

Upshot: a CLF and a smooth stabilizing controller exist.

Consider: $\dot{x} = -x^3 + u \longrightarrow \begin{array}{l} \dot{x} = -x^3 + \xi \\ \dot{\xi} = u \end{array}$ x, ξ, u scalar

main idea: the new system is stabilizable, and explicit CLF and controller can be given.

Integrator Backstepping: General Procedure

Base system: $\dot{x} = f(x) + G(x)u$ $\quad x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

Let a CLF $V_0(x)$ and a cont. stabilizing controller $u = k_0(x)$ be given.

Augmented system: $\begin{array}{l} \dot{x} = f(x) + G(x)\xi \\ \dot{\xi} = u \end{array}$ $\quad (x,\xi)$ – state, u – input

Goal: construct a CLF $V_1(x,\xi)$ and a stabilizing controller $u = k_1(x,\xi)$

New state space: $\mathbb{R}^n \times \mathbb{R}^m$
input (control) space: $\mathbb{R}^m$

Additional Assumption: $k_0 : \mathbb{R}^n \to \mathbb{R}^m$ is C^1

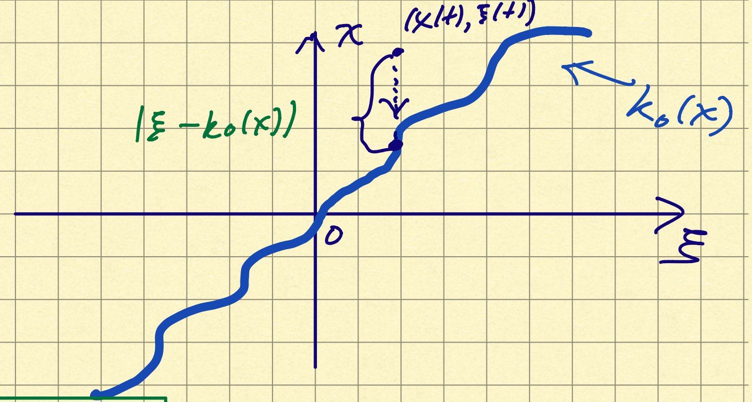
$\frac{\partial k_0}{\partial x}$: Jacobian $\quad \frac{\partial k_0}{\partial x}(x) \in \mathbb{R}^{m \times n}$

$$\left.\begin{array}{l} \dot{x} = f(x) + G(x)\xi \\ \dot{\xi} = u \end{array}\right\} \text{ new system}$$

If $\xi = k_0(x)$, then $\frac{\partial V_0}{\partial x} f(x,\xi) = \frac{\partial V_0}{\partial x} f(x, k_0(x)) < 0$

(since V_0 is a CLF, k_0 is a stab. controller for base system)

$$\mathcal{S} := \{(x,\xi) : \xi = k_0(x)\}$$



$$(x,\xi) \in \mathcal{S} \Rightarrow \frac{\partial V_0}{\partial x} f(x,\xi) < 0$$

This suggests:

$$\boxed{V_1(x,\xi) = V_0(x) + \frac{1}{2}|\xi - k_0(x)|^2}$$

Strategy:

- write down $\dot{V}_1$ [for unspecified $u = k_1(x,\xi)$]
- use CLF property of V_0 and stab. property of $k_0(x)$ to design $k_1(x,\xi)$ [and ensure V_1 is a CLF]

$$\dot{V}_1 = \frac{\partial V_1}{\partial x}\dot{x} + \frac{\partial V_1}{\partial \xi}\dot{\xi} \qquad \begin{aligned} \dot{x} &= f(x) + G(x)\xi \\ \dot{\xi} &= u \end{aligned}$$

$$\frac{\partial V_1}{\partial x} = \frac{\partial V_0}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{2}|\xi - k_0(x)|^2\right)$$

$$= \frac{\partial V_0}{\partial x} - \underbrace{(\xi - k_0(x))^T}_{\in \mathbb{R}^m} \underbrace{\frac{\partial k_0}{\partial x}}_{\mathbb{R}^{m\times n}} \qquad \frac{\partial V_1}{\partial x}(x) \in \mathbb{R}^{1\times n}$$

$$\frac{\partial V_1}{\partial \xi} = (\xi - k_0(x))^T$$

Important Reminder:

$V : \mathbb{R}^n \to \mathbb{R}$

$\frac{\partial V}{\partial x} \in \mathbb{R}^{1\times n}$

$F : \mathbb{R}^n \to \mathbb{R}^m \quad C^1$

$$\frac{\partial F}{\partial x}(x) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{m\times n}$$

$$\frac{\partial V_1}{\partial x} = \frac{\partial V_0}{\partial x} - (\xi - k_0(x))^T \frac{\partial k_0}{\partial x}$$

$$\frac{\partial V_1}{\partial \xi} = (\xi - k_0(x))^T$$

$$\dot V_1(x,\xi) = \frac{\partial V_1}{\partial x}\underbrace{\left(f(x) + G(x)\xi\right)}_{=\dot x} + \frac{\partial V_1}{\partial \xi}\underbrace{u}_{=\dot\xi}$$

$$= \left(\frac{\partial V_0}{\partial x} - (\xi - k_0(x))^T \frac{\partial k_0}{\partial x}\right)\left(f(x) + G\xi\right) + (\xi - k_0(x))^T u$$

$$= \underbrace{\frac{\partial V_0}{\partial x}\left(f(x) + G(x)\xi\right)}_{A} + \underbrace{(\xi - k_0(x))^T\left(u - \frac{\partial k_0}{\partial x}\left(f(x) + G\xi\right)\right)}_{B \text{ (involves } u)}$$

$$A = \underbrace{\frac{\partial V_0}{\partial x}\left(f(x) + G(x)k_0(x)\right)}_{\text{base system}} + \underbrace{\frac{\partial V_0}{\partial x} G(x)\left(\xi - k_0(x)\right)}_{\text{add + subtract}}$$

$$= \underbrace{\frac{\partial V_0}{\partial x}\left(f(x) + G(x)k_0(x)\right)}_{<0 \text{ for } x \neq 0} + (\xi - k_0(x))^T G(x)^T \frac{\partial V_0}{\partial x}^T$$

$$\Rightarrow \dot V_1 = \underbrace{\frac{\partial V_0}{\partial x}\left(f(x) + G(x)k_0(x)\right)}_{A}$$

$$+ (\xi - k_0(x))^T \underbrace{\left[u - \frac{\partial k_0}{\partial x}\left(f(x) + G(x)\xi\right) + G(x)^T \frac{\partial V_0^T}{\partial x}\right]}_{B}$$

$A < 0$ for $x \neq 0$ (by CLF property for (V_0, k_0))

$\dot V_1(x,\xi) < 0$ when $(x,\xi) \neq (0,0)$

Then we can take:

$$u = \underbrace{\frac{\partial k_0}{\partial x}\left(f(x) + G(x)\xi\right) - G(x)^T \frac{\partial V_0^T}{\partial x} + k_0(x) - \xi}_{:= k_1(x,\xi)}$$

If we take $u = k_1(x,\xi)$, then

$$\dot V_1(x,\xi) = \frac{\partial V_0}{\partial x}^T\left(f(x) + G(x)k_0(x)\right) - |\xi - k_0(x)|^2$$

$$< 0 \quad \text{when} \quad (x,\xi) \neq (0,0).$$

Summary:

$\dot{x} = f(x) + G(x)u$ $\longrightarrow$ $\dot{x} = f(x) + G(x)\xi$, $\dot{\xi} = u$

$V_0(x)$ - CLF

$u = k_0(x)$ - stab. controller

k_0 differentiable

$V_1(x,\xi) = V_0(x) + \frac{1}{2}|\xi - k_0(x)|^2$ - CLF

$u = k_1(x,\xi)$ - stab. controller

(involves $\frac{\partial k_0}{\partial x}$)

Back to example:

$$\left.\begin{aligned} \dot{x} &= -x^3 + \xi \\ \dot{\xi} &= u \end{aligned}\right\}$$ given 2-d system $(x,\xi) \in \mathbb{R}^2$, $u \in \mathbb{R}$

Step 1: base system

$$\dot{x} = -x^3 + u$$

$$V_0(x) = \frac{1}{2}x^2$$

$u = -x$ $\qquad \dot{V}_0 = -x^2 - x^4 < 0$

Step 2: augmented candidate CLF

$$V_1(x) = V_0(x) + \frac{1}{2}|\xi - k_0(x)|^2 \qquad k_0(x) = -x$$

$$= \frac{1}{2}x^2 + \frac{1}{2}(\xi + x)^2$$

$$\dot{V}_1(x,\xi) = \frac{\partial V_1}{\partial x}\dot{x} + \frac{\partial V_1}{\partial \xi}\dot{\xi}$$

$$= (2x + \xi)(-x^3 + \xi) + \overbrace{(\xi + x)}^{\xi - k_0(x)}u$$

$$= x(-x^3 + \xi) + (\xi + x)(-x^3 + \xi) + (\xi + x)u$$

$$= x(-x^3 + \xi) + (\xi + x)(u - x^3 + \xi)$$

$$= x(-x^3 - x) + x(\xi + x) + (\xi + x)(u - x^3 + \xi)$$

$$= \underbrace{-x^2 - x^4}_{\dot{V}_0(x)} + (\xi + x)(u - x^3 + x + \xi)$$

Choose u s.t. $u - x^3 + x + \xi = -(x + \xi)$

$$\Rightarrow \boxed{u = k_1(x, \xi) = x^3 - 2(x + \xi)}$$

$$\dot{V}_1(x, \xi) = -x^2 - x^4 - (x+\xi)^2 < 0 \quad \text{for } (x,\xi) \neq (0,0)$$

Backstepping can be iterated:

$$\dot{x} = f(x) + G(x)\xi_1$$

$$\dot{\xi}_1 = \xi_2$$

$$\vdots$$

$$\dot{\xi}_{k-1} = \xi_k$$

$$\dot{\xi}_k = u$$

assuming $\dot{x} = f(x) + G(x)u$ has a CLF V_0 and a C^k stab. control $k_0(x)$, we can iteratively construct

$V_0 \longrightarrow V_1 \longrightarrow V_2 \to \cdots \to V_k(x, \xi_1, \ldots, \xi_k)$ is a CLF and $u = k(x, \xi_1, \ldots, \xi_k)$ is a cont. stab. controller.

This procedure works more generally:

$$\dot{x} = f(x,u) \quad \longrightarrow \quad \begin{aligned} \dot{x} &= f(x,\xi) \\ \dot{\xi} &= h(x,\xi) + \underbrace{g(x,\xi)}_{\neq 0}u \end{aligned}$$

Backstepping vs. feedback linearization

$\dot{x} = f(x,u)$ is feedback linearizable if

$\exists$ an invertible, smooth coordinate change $x \mapsto z$ and a feedback controller $k(z)$ s.t. the closed-loop system

$$\dot{z} = \tilde{f}(z, k(z))$$

is linear and stable.

$z = \varphi(x) \qquad x = \varphi^{-1}(z)$

$$\dot z = \frac{d}{dt}\varphi(x) = \frac{\partial\varphi}{\partial x}\dot x = \frac{\partial\varphi}{\partial x} f(x,u)$$

$$= \underbrace{\frac{\partial\varphi}{\partial x}(\varphi^{-1}(z))\, f(\varphi^{-1}(z),u)}_{\tilde f(z,u)}$$

Ex.: $\quad \dot x = -x^3 + \xi$ $\qquad$ take $y = x$, $z = -x^3 + \xi$

$\dot\xi = u$

$\dot y = z$

$\dot z = -3x^2\dot x + \dot\xi$

$= -3x^2(-x^3 + \xi) + u$

$= -3x^5 - 3x^2\xi + u$

take $u = 3x^5 - 3x^2\xi - x - z$

Closed-loop system: $\quad \dot y = z$

$\dot z = -y - z$