Lyapunov-Based Design (cont.)

$\dot{x} = \theta x + u \qquad x, u \in \mathbb{R}$

$\left.\begin{array}{l} \dot{\vec{\theta}} = f(\vec{\theta}, x) \\ u = k(\vec{\theta}, x) \end{array}\right\}$ design f, k to achieve universal regulation

Candidate LF: $V(x, \vec{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\vec{\theta} - \theta)^2$

$$\dot{V} = \frac{\partial V}{\partial x}^T \dot{x} + \frac{\partial V}{\partial \vec{\theta}} \dot{\vec{\theta}}$$

$$= x(\theta x + \underbrace{k(\vec{\theta}, x)}_{=u}) + (\vec{\theta} - \theta)\underbrace{f(\vec{\theta}, x)}_{=\dot{\vec{\theta}}}$$

$$= \underbrace{\theta(x^2 - f(\vec{\theta}, x))}_{\text{want to set to } 0} + x k(\vec{\theta}, x) - \vec{\theta} f(\vec{\theta}, x)$$

$\Rightarrow \boxed{f(\vec{\theta}, x) = x^2}$

$\dot{\vec{\theta}} = x^2$

$\Rightarrow \dot{V} = x k(\vec{\theta}, x) - \vec{\theta} x^2 \quad = -x^2 < 0$ for $x \neq 0$

$\Rightarrow \boxed{k(\vec{\theta}, x) = -(\vec{\theta} + 1)x}$

General procedure ?

- Control Lyapunov fcn

$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$

$f(0, 0) = 0$

$V : \mathbb{R}^n \to [0, \infty)$ (usual regularity conditions) is a Control Lyapunov Function (CLF) if

$\forall x \neq 0 \quad \exists u \in \mathbb{R}^m$ s.t. $\frac{\partial V}{\partial x} f(x, u) < 0$

$\Longleftrightarrow \inf_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x} f(x, u) \right\} < 0 \qquad \forall x \neq 0$

Question: does there exist a continuous stabilizing state feedback law $u = k(x)$ s.t.

$$\frac{\partial V}{\partial x} f(x, k(x)) < 0 \qquad \text{for all } x \neq 0$$

and $k(0) = 0$.

CLF does not imply existence of such a $k(x)$ in general.

However, it does imply sol'n to continuous selection problem for systems affine in control

$$\dot{x} = f(x) + G(x)u \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ (drift)

$G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ $\qquad G(x) = [g_1(x) \vdots \cdots \vdots g_m(x)]$

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

CLF $\Rightarrow \exists$ cont. stabilizing feedback $u = k(x)$

Let $V : \mathbb{R}^n \to [0, \infty)$ be given

$$\dot{V}(x,u) = \frac{\partial V}{\partial x}\left[f(x) + \sum_{i=1}^{m} g_i(x) u_i\right]$$

$$= \frac{\partial V}{\partial x} f(x) + \sum_{i=1}^{m}\left(\frac{\partial V}{\partial x} g_i(x)\right) u_i$$

$$= a(x) + b(x)^T u$$

$b(x) = 0$ for some $x \neq 0$

$\Rightarrow \dot{V} = a(x)$

where $a(x) := \frac{\partial V}{\partial x} f(x)$,

$b(x) := \left(\frac{\partial V}{\partial x} g_1(x), \ldots, \frac{\partial V}{\partial x} g_m(x)\right)^T$

Claim: V is a CLF iff $b(x) = 0 \Rightarrow a(x) < 0$ for all $x \neq 0$.

(Claim was proved in Lec. 6)

Sontag's Universal Controller (E. Sontag, 1989)

Assume V is a CLF for $\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$.
Then the following controller solves the cont. selection problem:

$$u = k(x) = K(a(x), b(x)), \text{ where}$$

$$K(a,b) := \begin{cases} -\dfrac{a + \sqrt{a^2 + |b|^4}}{|b|^2}\, b, & b \neq 0 \\ 0, & b = 0 \end{cases}$$

where $|b|^2 := \sum_{i=1}^{m} |b_i|^2$.

Motivation:

$$\dot{V}(x,u) = a(x) + b(x)^T u$$

Suppose $\exists\, k(x)$ (cont.) s.t. $a(x) + b(x)^T k(x) < 0$ for $x \neq 0$

Consider a fictitious LTI system (x fixed)

$$\dot{z} = a(x) z + b(x)^T v \quad \text{with } z \in \mathbb{R},\ v \in \mathbb{R}^m$$

Then state feedback $v = k(x) z$ is stabilizing.

$$\tilde{V}(z) = \frac{1}{2} z^2$$

$$\dot{\tilde{V}} = (a(x) z + b(x)^T k(x) z) z$$

$$= \underbrace{(a(x) + b(x)^T k(x))}_{<0} z^2$$

$\Rightarrow$ Any $k(x)$ s.t. $a(x) + b(x)^T k(x) < 0$ is gives stabilizing controller $v = k(x) z$ for $\dot{z} = a(x) z + b(x)^T v$

Consider the following problem:

$$\min_{k(\cdot)} \int_0^\infty \left(|b(x)|^2 z^2(t) + v^2(t) \right) dt$$

Note: when $|b(x)| \approx 0$, v has to be small (small control property)

LQR (Linear Quadratic Regulator) problem:

- solve Algebraic Riccati Equation (ARE)

$$|b(x)|^2 p^2 - 2a(x)p - |b(x)|^2 = 0$$

for $p > 0$

Then the optimal stab. controller is

$$v = \underbrace{-pb(x)}_{k(x)} z$$

Solve the ARE (when $|b(x)| \neq 0$):

$$p = \frac{a(x) + \sqrt{a^2(x) + |b(x)|^4}}{|b(x)|^2}.$$

Example: $\dot{x} = -x^3 + u \qquad x, u \in \mathbb{R}$

construct a cont. stabilizing feedback law

Several options:

$V(x) = \frac{1}{2}x^2$

— $u = x^3 - x$ (feedback linearization)

$\dot{x} = -x \qquad \dot{V} = -x^2 < 0$

However: large control effort is needed when $|x|$ is large

— $u \equiv 0$ $\quad \dot{x} = -x^3$ $\quad \dot{V} = -x^4 < 0$

No control effort!

However: slow convergence to eq. when $x \approx 0$

— $u = -x$ $\quad \dot{x} = -x^3 - x$ $\quad \dot{V} = -x^2 - x^4 < 0$

— universal controller: $\dot{x} = -x^3 + u$ $\quad f(x) = -x^3$, $G(x) = 1$

$V(x) = \frac{1}{2}x^2 \qquad \frac{\partial V}{\partial x} = x$

$$a(x) = \frac{\partial V}{\partial x} f(x) = -x^4$$

$$b(x) = \frac{\partial V}{\partial x} G(x) = x$$

Is $V(x) = \frac{1}{2}x^2$ a CLF?

$$\begin{cases} \forall x \neq 0: \\ b(x) = 0 \Rightarrow a(x) < 0 \\ b(x) = x = 0 \text{ iff } x = 0 \\ a(x) = -x^4 < 0 \text{ for } x \neq 0 \end{cases}$$

$$u = k(x) = -\frac{-x^4 + \sqrt{x^8 + x^4}}{x^2} x \qquad (x \neq 0)$$

$$= -\frac{-x^4 + x^2\sqrt{x^4+1}}{x}$$

$$= +x^3 - x\sqrt{x^4+1} \qquad (\text{for all } x)$$

$k(0) = 0$

Closed-loop system: $\dot{x} = -\cancel{x^3} + \cancel{x^3} - x\sqrt{x^4+1}$

$$\dot{x} = -x\sqrt{x^4+1}$$

$k(x) = x^3 - x\sqrt{x^4+1}$:

$|x| \to \infty \qquad k(x) \to 0$

$|x| \to 0 \qquad k(x) \sim -x$

- universal controller interpolates between $u = -x$ (for small x) and $u = 0$ (for large x).

Back to our example:

$$\dot{x} = \theta x + u$$

$$\dot{\hat{\theta}} = x^2 \qquad V(x, \hat{\theta}) = \frac{1}{2}x^2 + \frac{1}{2}(\hat{\theta} - \theta)^2$$

— $\dot{V}$ is indep. of θ

Is $V(x, \hat{\theta})$ a CLF?

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{\theta}} \end{pmatrix} = \begin{pmatrix} \theta x \\ x^2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \qquad \text{— control-affine}$$

$$\dot{V}(x, \hat{\theta}) = x(\theta x + u) + (\hat{\theta} - \theta)x^2$$

$$= \hat{\theta}x^2 + xu$$

$$\Rightarrow \begin{cases} a(x, \hat{\theta}) = \hat{\theta}x^2 \\ b(x, \hat{\theta}) = x \end{cases}$$

For CLF: $(x, \hat{\theta}) \neq (0, \theta)$,

$$b(x, \hat{\theta}) = 0 \Rightarrow a(x, \hat{\theta}) < 0$$

Note: $b(x, \hat{\theta}) = 0$ for $x = 0$, $\hat{\theta}$ arbitrary

$a(x, \hat{\theta}) = \hat{\theta}x^2 \equiv 0$

— "weak" CLF

Apply the formula anyway: for $x \neq 0$,

$$u = k(x, \hat{\theta}) = -\frac{\hat{\theta}x^2 + \sqrt{\hat{\theta}^2 x^4 + x^4}}{x^2} x$$

$$= -\hat{\theta}x - \sqrt{\hat{\theta}^2 + 1}\, x$$

$$\equiv -(\hat{\theta} + \sqrt{\hat{\theta}^2 + 1})x$$

Closed-loop system:

$$\dot{x} = \theta x - (\hat{\theta} + \sqrt{\hat{\theta}^2 + 1})x$$

$$\dot{\hat{\theta}} = x^2$$

$$\dot{V} = \tilde{\theta} x^2 + x k(x, \hat{\theta})$$
$$= -\sqrt{\hat{\theta}^2 + 1}\, x^2 < 0 \quad \blacksquare$$