Review: Weak Lyapunov Functions

\[ \dot{x} = f(x) \quad f \text{ cont., } f(0) = 0 \]
\[ V(x) : \text{candidate Lyapunov funct} \]
\[ C, \quad V(x) \geq 0 \]

**Thm (weak lyapunov criterion)**

Suppose that \( \dot{V}(x) = V(x)^T f(x) \) satisfies
\[ \dot{V}(x) \leq -W(x), \quad \forall x \]
for some cont. fun \( W \) taking nonnegative values. Then, for every bounded trajectory \( x(t) \),
\[ W(x(t)) \to 0 \quad \text{as } t \to \infty. \]

**Reminder:** along the way, we used (and proved)
Barbalat’s lemma:

If \( x(t), x(t) \) remain bounded and
\[ \int_0^\infty W(x(t)) \, dt \]
exists and is finite, then \( W(x(t)) \to 0 \quad \text{as } t \to \infty. \)

**Connection to observability**

LTI system:
\[ x = Ax \quad x(t) \in \mathbb{R}^n \]
\[ A \in \mathbb{R}^{n \times n} \]

\[ V(x) = x^T P x \quad - \text{candidate LF} \]
\[ P = P^T > 0 \quad u^T P u > 0 \quad \forall u \]

\[ \dot{V}(x) = x^T (PA + A^T P)x \]
\[ \left[ \dot{V}(x(t)) = \frac{d}{dt} V(x(t)) = \frac{d}{dt} \frac{1}{2} x(t)^T P x(t) \right] \]

Assume \( \exists a \text{ matrix } C \) s.t.
\[ PA + A^T P \leq -C^T C \]

Consider a fictitious output \( y = Cx \)
\[
\begin{align*}
\dot{x} &= Ax \\
y &= Cx
\end{align*}
\]
If \((A, C)\) is an observable pair, then
\[
y(t) \to 0 \text{ as } t \to \infty \implies x(t) \to 0 \text{ as } t \to \infty.
\]
Back to Thm on weak Lyapunov funs:
\[
W(x) = x^T C^T C x
\]
\[
\dot{V}(x) \leq -W(x)
\]
x(t) remains bdd (since \(V(x)\) is p.d. and radially unbounded), so
\[
W(x(t)) \to 0 \text{ as } t \to \infty
\]
\[
y(t) \to 0 \text{ as } t \to \infty
\]
\[
x(t) \to 0 \text{ as } t \to \infty \text{ by observability.}
\]
This can be extended to LTV (linear time-varying systems) using Uniform Complete Observability \([\text{Kalman, 1960}]\).

Back to our example:
\[
\begin{align*}
\dot{x} &= \Theta x + u \\
\text{goal: } x(t) &\to 0 \text{ as } t \to \infty \\
\text{ult: } x(t) &\text{ remains bdd } \forall t \geq 0
\end{align*}
\]
\[
\begin{pmatrix}
\dot{x} \\
\dot{\Theta}
\end{pmatrix} =
\begin{pmatrix}
(x^2) \\
x^2
\end{pmatrix}
\]
\[
V(x, \Theta) = \frac{x^2}{2} + \frac{(\Theta - \Theta_0)^2}{2}
\]
Tuning law: \( \dot{\Theta} = x^2 \)
\[
u = -x(\Theta + 1)x
\]
Using weak Lyapunov fans, we showed:
\[ x(t) \to 0 \text{ as } t \to \infty \]
and \( \theta(t) \) remains bold \[ \Rightarrow \theta(t) \text{ remains bold} \]

Important: cannot guarantee \( \theta(t) \to \theta \) as \( t \to \infty \).

Recall our first attempt at analysis:
\[
V(x) = \frac{1}{2} x^2
\]
\[
\dot{V}(x) = x \dot{x}
\]
\[
= (\theta - \dot{\theta} - 1) x^2
\]

We will now show that adaptive regulation can be proved even for this choice of \( V \):
\[
\dot{V}(x) = (\theta - \dot{\theta} - 1) \dot{\theta}
\]

since \( \dot{\theta} = x^2 \)

\[
\dot{V}(x(t)) = (\theta - \theta(t) - 1) \dot{\theta}(t)
\]

Integrate: from 0 to \( t \)

LHS:
\[
\int_0^t \dot{V}(x(s)) \, ds = \int_0^t \frac{d}{ds} V(x(s)) \, ds = V(x(t)) - V(x(0))
\]

RHS:
\[
\int_0^t (\theta - \theta(s) - 1) \dot{\theta}(s) \, ds
\]
\[
= \int_0^t (\theta - 1) \dot{\theta}(s) \, ds - \int_0^t \dot{\theta}(s) \dot{\theta}(s) \, ds
\]
\[
\overset{1}{=} (\theta - 1) \left[ \theta(t) - \theta(0) \right] = (\theta - 1) \theta(t) + C_1
\]
\[
\overset{2}{=} \frac{1}{2} \dot{\theta}^2(s) \bigg|_0^t = \frac{1}{2} \dot{\theta}^2(t) + C_2
\]
\[
\overset{\text{depends only on } \dot{\theta}(0)}{=} \frac{1}{2} \dot{\theta}^2(t) + C_2
\]
\[
\overset{\text{depends only on } \theta(0)}{=} \frac{1}{2} \dot{\theta}^2(t) + C_2
\]
\[ V(x(t)) = (\theta - 1)x(t) - \frac{1}{4} \overline{\theta}^2(t) + C \]

\[ \frac{x^2(t)}{2} = (\theta - 1)x(t) - \frac{1}{4} \overline{\theta}^2(t) + C \]

depends only on \( x(0), \overline{\theta}(0) \)

Some observations:
1) \( \text{LHS} \geq 0 \implies \text{RHS} \geq 0 \) for all \( t \geq 0 \)
2) \( \text{RHS} \) (as a function of \( \overline{\theta}(t) \)) is a concave quadratic function

RHS
\[ \begin{array}{c}
\text{RHS} \\
\overline{\theta} \\
\end{array} \]

Thus: \( \overline{\theta}(t) \) remains bound \( \forall t \geq 0 \)

In detail: \( \overline{\theta}(t) = x^2(t) \implies \overline{\theta}(t) \) is monotone nondecreasing

\( \implies \) so \( \overline{\theta}(t) \) either grows w/o bound or remains bounded

But if \( \overline{\theta}(t) \) grows w/o bound, then eventually \( \text{RHS} \) will be negative, which cannot happen.

\[ x^2(t) = 2 \{ (\theta - 1)\overline{\theta}(t) - \frac{1}{4} \overline{\theta}^2(t) \} + C^2 \]

\[ \sup_{t \geq 0} x^2(t) = 2 \sup_{t \geq 0} \{ (\theta - 1)\overline{\theta}(t) - \frac{1}{4} \overline{\theta}^2(t) \} + C^2 \leq 0 \]

\( \overline{\theta}(t) \) remains bounded, \( x(t) \) remains bounded

\[ |x(t)| = |(\theta - \overline{\theta}(t) - 1)x(t)| \] remains bounded

\[ 0 \leq \int_0^t x^2(s) \, ds = \int_0^t \overline{\theta}^2(s) \, ds = \overline{\theta}(t) - \overline{\theta}(0) \] remains bounded
\[ \int_0^\infty x^2(t) \, dt < \infty \]

So, \( x(t) \to 0 \) as \( t \to \infty \) by Barbalat's lemma.

**Main Takeaways:**

- **the control law**
  \[
  \begin{cases}
  u = -k(\theta)x \\
  \dot{\theta} = x^2
  \end{cases}
  \quad (k(\theta) = \theta + 1)
  \]

  is capable of stabilizing any plant of the form
  \[
  \dot{x} = \theta x + u,
  \quad \theta \in \mathbb{R}.
  \]

- Moreover, all signals in the closed-loop system remain bounded for all \( t > 0 \), \( x \) arbitrary initial condition.

  The above controller is a **universal regulator** for the class
  \[
  \left\{ \dot{x} = \theta x + u : \theta \in \mathbb{R} \right\}.
  \]

- The notation \( \theta \) suggests estimating what we are doing is rather exhaustive search through the space of controller gains until we find something that works.

**Downside:** possibly poor transient behavior.

**Universal Regulators for Scalar Plants**

Consider the class of plants
\[
\left\{ y = ay + bu : a \in \mathbb{R}, \quad b > 0 \right\}
\]
(Before: \( a \in \mathbb{R}, \ b = 0 \))

Do universal regulators exist?

**Preview:** if \( \text{sign}(b) \) is unknown, a universal regulator exists, but it is not simple.
Simple case: \( \text{sign}(b) \) known

[Suppose \( b > 0 \), without loss of generality]

Claim: \( u = -ky \), \( k = y^2 \in \text{tuning law} \)

Closed-loop system:
\[
\dot{y} = (a - bk)y \\
k = y^2
\]

\( \mathcal{U}(y) = \frac{y^2}{2} \) [Lyaupunov func]

\[
\dot{\mathcal{U}}(y) = y \dot{y} \quad \text{--- that is, } \frac{d}{dt} \mathcal{U}(y(t)) = (a - bk(t))y^2(t)
\]

Integrate to get:
\[
\frac{y^2(t)}{2} = ak(t) - \frac{1}{2}k^2(t) + C
\]

--- same argument as earlier gives:

\[
\sup_{t \geq 0} y^2(t) = 2 \sup_{t \geq 0} \left\{ ak(t) - \frac{1}{2}k^2(t) + C \right\} < \infty
\]

\( k(t) \in L^\infty \Rightarrow y(t) \in L^\infty \)

\[
y(t) = (a - bk(t))y(t) \quad \text{in} \quad L^\infty
\]

\[
0 \leq \int_0^t y^2(s)ds \quad \text{in} \quad L^\infty \Rightarrow \int_0^\infty y^2(t)dt < \infty
\]

\( \Rightarrow y(t) \to 0 \text{ as } t \to \infty \) by Barbalat’s lemma.