ECE 563: Information Theory (Fall 2017)

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Homework 6

Assigned November 2, 2017; due November 9, 2017

Reading:

- Polyanskiy and Wu, Sections 16.1–16.4, 17.1–17.4, 18.1–18.4.

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1. A spy sends messages to his contact as follows. Each hour he either does not telephone, or he telephones and allows the telephone to ring a certain number of times – not more than N, for fear of detection. His contact does not answer, but merely notes whether or not the telephone rings, and, if so, how many times. Because of deficiencies in the telephone system, calls may fail to be properly connected; the correct connection has probability p, where $0 < p < 1$, and is independent for distinct calls, but the spy has no means of knowing which calls reach his contact. If the connection is made, then the number of rings is transmitted correctly. The probability of a false connection from another subscriber at a time when no call is made may be neglected.

   (a) Write down the transition probabilities for this channel and calculate the capacity explicitly.

   (b) Determine a condition on N in terms of p which will imply, with optimal coding, that the spy will always telephone.

2. In this problem, we focus on the binary erasure channel (BEC). Consider a code with $M = 2^k$ messages operating over a blocklength-n BEC with erasure probability $\delta \in [0,1)$.

   (a) Prove that, regardless of the encoder/decoder pair,

   $$\mathbf{P}[\text{error}|\#\text{ erasures} = z] \geq (1 - 2^{n-z-k})^+.$$

   (b) Conclude, by averaging with respect to the distribution of the number of erasures, that the probability of error ε must satisfy

   $$\varepsilon \geq \sum_{\ell=n-k+1}^{n} \binom{n}{\ell} \delta^\ell (1-\delta)^{n-\ell} (1 - 2^{n-\ell-k}).$$

(c) By applying the DT bound with uniform P_X, show that there exist codes with

$$\varepsilon \le \sum_{t=0}^{n} \binom{n}{t} \delta^t (1-\delta)^{n-t} 2^{-(n-t-k+1)^+}.$$

(d) Fix $n = 500, \delta = 1/2$. Compute the smallest k, for which the right-hand side of the converse bound in part (b) is greater than 10^{-3}.

(e) For the same n and δ, find the largest k, for which the right-hand side of the achievability bound in part (c) is smaller than 10^{-3}.

(f) Express your results in terms of lower and upper bounds on $\log M^*(500, 10^{-3})$.

3. Recall that, in the proof of the DT bound, we have used the decoder that outputs (for a given y) the smallest index m that satisfies $i(c_m; y) > \log\gamma$, for some choice of the threshold γ.

   Consider the following generalization. Fix a subset $E \subset \mathcal{X} \times \mathcal{Y}$, and let the decoder output the first m, such that $(c_m, y) \in E$. Modify the proof of the DT bound, show that the average probability of error with this decoder satisfies

$$\mathbf{E}[P_e] \le \mathbf{P}[(X,Y) \notin E] + \frac{M-1}{2}\mathbf{P}[(\bar{X}, Y) \in E].$$

   Conclude that the optimal region E is given by $E = \{(x,y) \in \mathcal{X} \times \mathcal{Y} : i(x;y) > \log \frac{M-1}{2}\}$.

4. In this problem, we will examine the tightness of the weak converse bounds.

   (a) Recall that the bit error rate P_b satisfies

$$\log M \le \frac{\sup_{P_X} I(X;Y)}{\log 2 - h_2(P_\mathrm{b})}.$$

   Let X and Y be the input and output sequences of $\mathrm{BSC}(n, \delta)$. Show that the bound above can be attained with equality
   Hint: Try $M = 2^n$.

   (b) Does there exist a combination of the channel $P_{Y|X}$, $\varepsilon > 0$, and an (M, ε)-code for $P_{Y|X}$ that attains the bound

$$\log M \le \frac{\sup_{P_X} I(X;Y) + h_2(\varepsilon)}{1-\varepsilon}$$

   with equality?

5. Let W_1 and W_2 denote the channel matrices of two discrete memoryless channels (DMC's) $P_{Y_1|X_1}$ and $P_{Y_2|X_2}$ with capacities C_1 and C_2, respectively. The *sum* of these channels is another DMC with channel matrix $W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}$. Show that the capacity of the sum channel is given by

$$C = \log\left(\exp(C_1) + \exp(C_2)\right).$$