ECE 563: Information Theory (Fall 2017)

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Homework 5

Assigned October 26, 2017; due November 2, 2017

Reading:

- Polyanskiy and Wu, Sections 12.1–12.5, 13.1–13.3, 14.1, 15.1–2.

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1. Let P_0 and P_1 be probability distributions on a common space $\mathcal{X}$. Recall the definition of the Neyman–Pearson region

$$\mathcal{R}(P_0,P_1) := \Big\{(\alpha,\beta) \in [0,1]^2 : \exists P_{Z|X} : \mathcal{X} \to \{0,1\} \text{ such that } \alpha = P_0[Z=0], \beta = P_1[Z=0]\Big\}.$$

   Also, let $P_{Y|X} : \mathcal{X} \to \mathcal{Y}$ be a random transformation that carries P_j into Q_j according to $P_j \xrightarrow{P_{Y|X}} Q_j$, $j \in \{0,1\}$. Compare the regions $\mathcal{R}(P_0,P_1)$ and $\mathcal{R}(Q_0,Q_1)$. What does this say about $\beta_\alpha(P_0,P_1)$ vs. $\beta_\alpha(Q_0,Q_1)$?

2. (a) Consider the binary hypothesis test

$$\begin{aligned} H_0 &: X \sim \mathcal{N}(0,1) \\ H_1 &: X \sim \mathcal{N}(\mu,1) \end{aligned}$$

   for some $\mu \neq 0$. Compute the Neyman–Pearson region $\mathcal{R}(\mathcal{N}(0,1),\mathcal{N}(\mu,1))$.

   (b) Now suppose that we have n i.i.d. samples, so we wish to test

$$\begin{aligned} H_0 &: X_1,\ldots,X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1) \\ H_1 &: X_1,\ldots,X_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu,1) \end{aligned}$$

   Compute the region $\mathcal{R}(\mathcal{N}(0,1)^{\otimes n},\mathcal{N}(\mu,1)^{\otimes n})$. Describe how the region evolves as $n \to \infty$ and provide an interpretation.
   Hint: Consider sufficient statistics.

3. Recall the definition of the total variation distance:

$$\mathrm{TV}(P,Q) := \sup_E |P[E] - Q[E]|.$$

(a) Prove that

$$\mathrm{TV}(P,Q) = \sup_{\alpha \in [0,1]} \left(\alpha - \beta_\alpha(P,Q)\right).$$

Explain how to read off the value of $\mathrm{TV}(P,Q)$ from the region $\mathcal{R}(P,Q)$.

(b) Consider a binary hypothesis test

$$\begin{aligned} H_0 &: X \sim P \\ H_1 &: X \sim Q \end{aligned}$$

Fix a prior $\pi = (\pi_0, \pi_1)$ such that $\pi_0 + \pi_1 = 1$ and $0 < \pi_0 < 1$. Denote the optimal Bayesian average probability of error by

$$P_e := \inf_{\text{tests } P_{Z|X}} \left(\pi_0 P[Z=1] + \pi_1 Q[Z=0]\right).$$

Prove that, for $\pi = \left(\frac{1}{2}, \frac{1}{2}\right)$,

$$P_e = \frac{1}{2}\left(1 - \mathrm{TV}(P,Q)\right).$$

Find the optimal test.

(c) Find the optimal test for a general prior π (not necessarily equiprobable).

(d) Why is it always sufficient to focus on deterministic tests in order to achieve the Bayesian optimum?

4. Consider the binary hypothesis testing problem from 2(b).

(a) Compute the Stein exponent.

(b) Compute the region $\mathcal{E}$ of achievable error exponent pairs (E_0, E_1). Express the optimal boundary in explicit form (eliminate the parameter).

(c) Identify the divergence-minimizing path $P^{(\lambda)}$ running from P to Q for $\lambda \in [0,1]$. Verify that $(E_0, E_1) = \left(D(P^{(\lambda)} \| P), D(P^{(\lambda)} \| Q)\right)$ for $\lambda \in [0,1]$ gives the same trade-off curve.

(d) Compute the Chernoff exponent.