ECE 563: Information Theory (Fall 2017)

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Homework 2

Assigned September 7, 2017; due September 14, 2017

Reading:

- Polyanskiy and Wu, Sections 2.1–2.5, 3.1

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1. In this problem, we will explore the interpretation of entropy as *asymptotic combinatorial complexity*.

   (a) Fix two integers $n \ge 1$ and $0 \le k \le n$, and let $T^n_k \subset \{0,1\}^n$ be the set of all binary sequences of length n that have exactly k ones, i.e.,

   $$T^n_k := \Big\{x^n = (x_1, \dots, x_n) \in \{0,1\}^n : |\{i : x_i = 1\}| = k\Big\}.$$

   Prove that, for $k \in [1, n-1]$,

   $$|T^n_k| = \sqrt{\frac{1}{k(1-\frac{k}{n})}}\exp\left(nh_2\left(\frac{k}{n}\right)\right)C(n,k),$$

   where $h_2(p) := -p\log p - \bar{p}\log\bar{p}$ is the binary entropy function, and $C(n,k)$ is bounded from above and below by two absolute constants independent of n and k, i.e., $C_0 \le C(n,k) \le C_1$. Conclude that, for all $0 \le k \le n$,

   $$\log|T^n_k| = nh_2\left(\frac{k}{n}\right) + O(\log n).$$

   Hint: Use Stirling's approximation

   $$e^{\frac{1}{12n+1}} \le \frac{n!}{\sqrt{2\pi n}(n/e)^n} \le e^{\frac{1}{12n}}, \qquad n \ge 1.$$

   (b) Let $Q^n = \mathrm{Bern}(q)^n$ be the i.i.d. Bernoulli distribution on $\{0,1\}^n$. Then $Q^n[T^n_k]$ is the probability to get exactly k ones. Prove that

   $$\log Q^n[T^n_k] = -nd\Big(\frac{k}{n}\Big\|q\Big) + O(\log n),$$

   where $d(p\|q) := p\log\frac{p}{q} + \bar{p}\log\frac{\bar{p}}{\bar{q}}$ is the binary divergence function.

2. Let X and Y be a pair of jointly distributed random variables with $I(X;Y) < \infty$. Let Z be obtained from Y by passing it through an *erasure channel* with erasure probability ε, i.e.,

$$Z = \begin{cases} Y, & \text{with probability } 1-\varepsilon \\ \square, & \text{with probability } \varepsilon \end{cases}$$

where $\square$ is an *erasure symbol* not in the alphabet of Y. Compute the mutual information $I(X;Z)$.

3. Let X be a random variable taking positive integer values, such that $\mathbf{E}X < \infty$. Prove that

$$H(X) \le \mathbf{E}X \cdot h_2\left(\frac{1}{\mathbf{E}X}\right).$$

Show that equality is achieved if and only if X is a geometric random variable.

Hint: Find a suitable probability distribution Q, such that RHS $-$ LHS $= D(P_X \| Q)$.

4. Let $Z_1, \ldots, Z_n$ be independent Poisson random variables with parameter λ. Prove that $S_n := \sum_{i=1}^n Z_i$ is a sufficient statistic of $Z^n = (Z_1, \ldots, Z_n)$ for λ.

5. Recall that the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in [0,1]$, denoted by $\text{Bin}(n,p)$, is given by

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k \in \{0, 1, \ldots, n\}.$$

Prove that

$$D(\text{Bin}(n,p) \| \text{Bin}(n,q)) = n d(p \| q).$$

Hint: Think about sufficient statistics.