Homework 5: Solutions

December 9, 2015

1 Killed Markov chains

(a)
- $\|A\|_\infty \geq 0$, with equality if and only if A is the zero matrix: Since $a_{ij} \leq \|A\|_\infty$ for all i, j, $\|A\|_\infty = 0 \implies A = 0$; the reverse direction and the fact that $\|A\|_\infty \geq 0$ are obvious.
- $\|cA\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^n |ca_{ij}| = |c|\|A\|_\infty$.
- $\|A+B\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^n |a_{ij}+b_{ij}| \leq \max_{1\leq i\leq m} \left(\sum_{j=1}^n |a_{ij}| + \sum_{j=1}^n |b_{ij}|\right) \leq \|A\|_\infty + \|B\|_\infty$.
- $\|A\|_\infty$ is submultiplicative:

$$\begin{aligned}
\|AB\|_\infty &= \max_{1\leq i\leq m} \sum_{j=1}^n \left|\sum_{k=1}^n A_{ik}B_{kj}\right| \leq \max_{1\leq i\leq m} \sum_{j=1}^n \sum_{k=1}^n |A_{ik}||B_{kj}| = \max_{1\leq i\leq m} \sum_{k=1}^n \left(|A_{ik}| \sum_{j=1}^n |B_{kj}|\right) \\
&\leq \max_{1\leq i\leq m} \sum_{k=1}^n \left(|A_{ik}| \max_{1\leq k\leq n} \sum_{j=1}^n |B_{kj}|\right) = \left(\max_{1\leq i\leq m} \sum_{k=1}^n |A_{ik}|\right)\left(\max_{1\leq k\leq n} \sum_{j=1}^n |B_{kj}|\right) \\
&= \|A\|_\infty \|B\|_\infty.
\end{aligned}$$

(b) Prove by induction.

Base case: $n = 1$. For any i, $\mathbb{P}[\zeta > 1|X_0 = i] = 1 - \mathbb{P}[\zeta = 1|X_0 = i] = \sum_{j=1}^{|\mathcal{S}|-1} a_{ij} \leq \|A\|_\infty$.

Suppose for some $k \geq 1$, $\max_{i\in\mathcal{A}} \mathbb{P}[\zeta > k|X_0 = i] \leq \|A\|_\infty^k$. Then for any i,

$$\begin{aligned}
\mathbb{P}[\zeta > k+1|X_0 = i] &= \sum_{j\in\mathcal{A}} \mathbb{P}[X_1 = j|X_0 = i]\mathbb{P}[\zeta > k+1|X_1 = j, X_0 = i] \\
\text{(time-homogeneous Markov chain)} &= \sum_{j\in\mathcal{A}} \mathbb{P}[X_1 = j|X_0 = i]\mathbb{P}[\zeta > k|X_0 = j] \\
\text{(by inductive assumption)} &\leq \sum_{j\in\mathcal{A}} \mathbb{P}[X_1 = j|X_0 = i]\|A\|_\infty^k \\
&\leq \|A\|_\infty^{k+1}.
\end{aligned}$$

Therefore, for all $n \geq 1$, $\max_{i\in\mathcal{A}} \mathbb{P}[\zeta > n|X_0 = i] \leq \|A\|_\infty^n$.

(c) If $\|A\|_\infty < 1$, then $\lim_{n\to\infty} \mathbb{P}[\zeta > n] = 0$, which implies that $\mathbb{P}[\zeta < \infty] = 1$. In this case, for any i,

$$\mathbb{E}[\zeta|X_0 = i] = \sum_{n=0}^\infty \mathbb{P}[\zeta > n|X_0 = i] \leq 1 + \|A\|_\infty + \|A\|_\infty^2 + \ldots = \frac{1}{1 - \|A\|_\infty}.$$

2 Towards a stochastic integral

(a) (i) Let $\mathcal{T}_1 = \{t_1^{(1)}, \dots, t_{n-1}^{(1)}\}$ and $\mathcal{T}_2 = \{t_1^{(2)}, \dots, t_{n-1}^{(2)}\}$ be the set of jumping times of f_1 and f_2, respectively. Define $\mathcal{T} = \{t_1, \dots, t_{m-1}\} \triangleq \mathcal{T}_1 \cup \mathcal{T}_2$ as the common refinement of $\mathcal{T}_1$ and $\mathcal{T}_2$. Then $c_1 f_1 + c_2 f_2$ is constant on each interval $[t_i, t_{i+1})$, and

$$\begin{aligned} I(c_1 f_1 + c_2 f_2) &= \sum_{i=0}^{m-1} (c_1 f_1(t_i) + c_2 f_2(t_i))(W_{t_{i+1}} - W_{t_i}) \\ &= \sum_{i=0}^{m-1} c_1 f_1(t_i)(W_{t_{i+1}} - W_{t_i}) + \sum_{i=0}^{m-1} c_2 f_2(t_i)(W_{t_{i+1}} - W_{t_i}) \\ &= c_1 I(f_1) + c_2 I(f_2). \end{aligned}$$

To show that $I(f)$, $f \in S(0,T)$ are jointly Gaussian, it suffices to show that any finite collection $I(f_1), \dots, I(f_k)$ are jointly Gaussian. Define $\mathcal{T} = \{t_1, \dots, t_{m-1}\} \triangleq \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_k$ as the common refinement of $\mathcal{T}_1, \dots, \mathcal{T}_k$, the set of jumping times of $f_1, \dots, f_k$. Then

$$I(f_j) = \sum_{i=0}^{m-1} f_j(t_i)(W_{t_{i+1}} - W_{t_i}), \qquad \text{for } j = 1, \dots, k.$$

From the independent increment property of the Wiener process, we know that $\tilde{W}_i \triangleq W_{t_{i+1}} - W_{t_i}$, $i = 0, \dots, m-1$, are independent and each has the distribution $N(0, t_{i+1} - t_i)$. Thus each of $I(f_1), \dots, I(f_k)$ can be viewed as a linear combination of m i.i.d. standard Gaussian random variables. This proves that $I(f_1), \dots, I(f_k)$ are jointly Gaussian.

(ii) Since A and B are disjoint, the independence of $I(\mathbf{1}_A)$ and $I(\mathbf{1}_B)$ follows from the independent increment property of the Wiener process.

(iii) Use the same definitions of $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}$, and $\tilde{W}_i$, $i = 0, \dots, m-1$, as in Part (a(i)). Then

$$\begin{aligned} \mathbb{E}[I(f_1)I(f_2)] &= \mathbb{E}\left[\left(\sum_{i=0}^{m-1} f_1(t_i)\tilde{W}_i\right)\left(\sum_{i=0}^{m-1} f_2(t_i)\tilde{W}_i\right)\right] \\ (\tilde{W}_i\text{'s are 0-mean and independent}) &= \sum_{i=0}^{m-1} f_1(t_i) f_2(t_i) \mathbb{E}[\tilde{W}_i^2] \\ (\tilde{W}_i \sim N(0, t_{i+1} - t_i)) &= \sum_{i=0}^{m-1} f_1(t_i) f_2(t_i)(t_{i+1} - t_i) \\ &= \int_0^T f_1(t) f_2(t) \mathrm{d}t. \end{aligned}$$

(iv) $\mathbb{E}[I(f)] = \sum_{i=0}^{n-1} f(t_i)\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$; from Part (a(i)), we know that $I(f)$ is Gaussian; from Part (a(iii)), we know that $\mathrm{Var}[I(f)] = \|f\|^2_{L^2(0,T)}$.

(b) Since the properties of step functions proved in Part (a) continue to hold for every $f \in L^2(0,T)$, we know that $I(\varphi_i) \sim N(0,1)$, and $\mathbb{E}[I(\varphi_i)I(\varphi_j)] = 0$ for $i \neq j$. Thus $I(\varphi_1), I(\varphi_2), \dots$ are i.i.d. $N(0,1)$.