1 Killed Markov chains

(a) \( \|A\|_\infty \geq 0 \), with equality if and only if \( A \) is the zero matrix: Since \( a_{ij} \leq \|A\|_\infty \) for all \( i, j \), \( \|A\|_\infty = 0 \iff A = 0 \); the reverse direction and the fact that \( \|A\|_\infty \geq 0 \) are obvious.

- \( \|cA\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |ca_{ij}| = |c|\|A\|_\infty \).
- \( \|A + B\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij} + b_{ij}| \leq \max_{1 \leq i \leq m} \left( \sum_{j=1}^{n} |a_{ij}| + \sum_{j=1}^{n} |b_{ij}| \right) \leq \|A\|_\infty + \|B\|_\infty \).
- \( \|A\|_\infty \) is submultiplicative:

\[
\|AB\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_{ik} B_{kj} \right| \leq \max_{1 \leq i \leq m} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_{ik} \right| \left| \sum_{j=1}^{n} B_{kj} \right| = \left( \max_{1 \leq i \leq m} \sum_{k=1}^{n} |A_{ik}| \right) \left( \max_{1 \leq k \leq n} \sum_{j=1}^{n} |B_{kj}| \right) = \|A\|_\infty \|B\|_\infty.
\]

(b) Prove by induction.

Base case: \( n = 1 \). For any \( i \), \( \mathbb{P}[\zeta > 1 | X_0 = i] = 1 - \mathbb{P}[\zeta = 1 | X_0 = i] = \sum_{j=1}^{|S|} a_{ij} \leq \|A\|_\infty \).

Suppose for some \( k \geq 1 \), \( \max_{i \in A} \mathbb{P}[\zeta > k | X_0 = i] \leq \|A\|_\infty^k \). Then for any \( i \),

\[
\mathbb{P}[\zeta > k + 1 | X_0 = i] = \sum_{j \in A} \mathbb{P}[X_1 = j | X_0 = i] \mathbb{P}[\zeta > k + 1 | X_1 = j, X_0 = i]
\]

(time-homogeneous Markov chain) \( = \sum_{j \in A} \mathbb{P}[X_1 = j | X_0 = i] \mathbb{P}[\zeta > k | X_0 = j] \)

(by inductive assumption) \( \leq \sum_{j \in A} \mathbb{P}[X_1 = j | X_0 = i] \|A\|_\infty^k \)

\( \leq \|A\|_\infty^{k+1} \).

Therefore, for all \( n \geq 1 \), \( \max_{i \in A} \mathbb{P}[\zeta > n | X_0 = i] \leq \|A\|_\infty^n \).

(c) If \( \|A\|_\infty < 1 \), then \( \lim_{n \to \infty} \mathbb{P}[\zeta > n] = 0 \), which implies that \( \mathbb{P}[\zeta < \infty] = 1 \). In this case, for any \( i \),

\[
\mathbb{E}[\zeta | X_0 = i] = \sum_{n=0}^{\infty} \mathbb{P}[\zeta > n | X_0 = i] \leq 1 + \|A\|_\infty + \|A\|_\infty^2 + \ldots = \frac{1}{1 - \|A\|_\infty}.
\]
2 Towards a stochastic integral

(a) (i) Let $T_1 = \{t_1^{(1)}, \ldots, t_n^{(1)}\}$ and $T_2 = \{t_1^{(2)}, \ldots, t_n^{(2)}\}$ be the set of jumping times of $f_1$ and $f_2$, respectively. Define $T = \{t_1, \ldots, t_{m-1}\} \triangleq T_1 \cup T_2$ as the common refinement of $T_1$ and $T_2$. Then $c_1f_1 + c_2f_2$ is constant on each interval $[t_i, t_{i+1})$, and

$$I(c_1f_1 + c_2f_2) = \sum_{i=0}^{m-1} (c_1f_1(t_i) + c_2f_2(t_i))(W_{t_{i+1}} - W_{t_i})$$

$$= \sum_{i=0}^{m-1} c_1f_1(t_i)(W_{t_{i+1}} - W_{t_i}) + \sum_{i=0}^{m-1} c_2f_2(t_i)(W_{t_{i+1}} - W_{t_i})$$

$$= c_1I(f_1) + c_2I(f_2).$$

To show that $I(f)$, $f \in S(0, T)$ are jointly Gaussian, it suffices to show that any finite collection $I(f_1), \ldots, I(f_k)$ are jointly Gaussian. Define $\mathcal{T} = \{t_1, \ldots, t_{m-1}\} \triangleq T_1 \cup \ldots \cup T_k$ as the common refinement of $T_1, \ldots, T_k$, the set of jumping times of $f_1, \ldots, f_k$. Then

$$I(f_j) = \sum_{i=0}^{m-1} f_j(t_i)(W_{t_{i+1}} - W_{t_i}), \quad \text{for } j = 1, \ldots, k.$$ 

From the independent increment property of the Wiener process, we know that $\tilde{W}_i \triangleq W_{t_{i+1}} - W_t$, $i = 0, \ldots, m-1$, are independent and each has the distribution $N(0, t_{i+1} - t_i)$. Thus each of $I(f_1), \ldots, I(f_k)$ can be viewed as a linear combination of $m$ i.i.d. standard Gaussian random variables. This proves that $I(f_1), \ldots, I(f_k)$ are jointly Gaussian.

(ii) Since $A$ and $B$ are disjoint, the independence of $I(1_A)$ and $I(1_B)$ follows from the independent increment property of the Wiener process.

(iii) Use the same definitions of $T_1$, $T_2$, $\mathcal{T}$, and $\tilde{W}_i$, $i = 0, \ldots, m-1$, as in Part (a(i)). Then

$$\mathbb{E}[I(f_1)I(f_2)] = \mathbb{E}\left[\left(\sum_{i=0}^{m-1} f_1(t_i)\tilde{W}_i\right)\left(\sum_{i=0}^{m-1} f_2(t_i)\tilde{W}_i\right)\right]$$

($\tilde{W}_i$'s are 0-mean and independent) \[\sum_{i=0}^{m-1} f_1(t_i)f_2(t_i)\mathbb{E}[\tilde{W}_i^2]\]

($\tilde{W}_i \sim N(0, t_{i+1} - t_i)$) \[\sum_{i=0}^{m-1} f_1(t_i)f_2(t_i)(t_{i+1} - t_i)\]

\[= \int_0^T f_1(t)f_2(t)dt.\]

(iv) $\mathbb{E}[I(f)] = \sum_{i=0}^{m-1} f(t_i)\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$; from Part (a(i)), we know that $I(f)$ is Gaussian; from Part (a(iii)), we know that $\text{Var}[I(f)] = \|f\|_2^2$.

(b) Since the properties of step functions proved in Part (a) continue to hold for every $f \in L^2(0, T)$, we know that $I(\varphi_i) \sim N(0, 1)$, and $\mathbb{E}[I(\varphi_i)I(\varphi_j)] = 0$ for $i \neq j$. Thus $I(\varphi_1), I(\varphi_2), \ldots$ are i.i.d. $N(0, 1)$. 

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