ECE 534 RANDOM PROCESSES **FALL 2015**
PROBLEM SET 5 **Due beginning of Class, Dec 8, for EXTRA CREDIT**

5. Markov Chains and Stochastic Integrals

Assigned reading: Chapter 6 (Sections 1–2), Chapter 7 (Sections 1–3, 6) of the ECE 534 course notes. Be sure to at least skim through the problems at the end of the chapters.

Problems to be handed in:

1 Killed Markov chains

A *killed Markov chain* is a discrete-time, finite-state, time-homogeneous Markov process with a distinguished *dead state* $\mathfrak{d}$, such that, for any t,

$$\mathbb{P}[X_{t+1} = \mathfrak{d} | X_t = \mathfrak{d}] = 1$$

— in other words, once the chain is dead, it stays dead. In this problem, we will analyze the long-term behavior of killed Markov chains and relate it to certain matrix norms.

(a) As a warm-up, we will need some facts about matrix norms. Let A be an $m \times n$ matrix with real-valued entries. Consider the following quantity:

$$\|A\|_\infty := \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|.$$

In other words, $\|A\|_\infty$ is the largest absolute row sum of the matrix. Prove the following:

(i) $\|A\|_\infty$ is a norm, i.e., $\|A\|_\infty \ge 0$, with equality if and only if A is the zero matrix; $\|cA\|_\infty = |c|\|A\|_\infty$ for any $c \in \mathbb{R}$; $\|A + B\|_\infty \le \|A\|_\infty + \|B\|_\infty$ for any two $m \times n$ matrices A and B.

(ii) $\|A\|_\infty$ is submultiplicative, i.e., if A and B are square matrices, then $\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty$.

(b) Consider a killed Markov chain with finite state space $\mathcal{S}$, and let $\mathcal{A} \triangleq \mathcal{S} \backslash \{\mathfrak{d}\}$ be the subset of its "alive" states. Let $A = (a_{ij})_{i,j \in \mathcal{A}}$ be the matrix of transition probabilities between the alive states: for all t,

$$a_{ij} = \mathbb{P}[X_{t+1} = j | X_t = i], \qquad \forall i, j \in \mathcal{A}.$$

Define the *killing time* $\zeta \triangleq \inf\{t \in \mathbb{N} : X_t = \mathfrak{d}\}$ (we assume that the chain starts out alive). Prove that

$$\max_{i \in \mathcal{A}} \mathbb{P}[\zeta > n | X_0 = i] \le \|A\|_\infty^n$$

(note: $\|A\|_\infty \le 1$, i.e., the matrix A is *substochastic*).

(c) Prove that if $\|A\|_\infty < 1$, then the chain dies in finite time with probability one, and that

$$\max_{i \in \mathcal{A}} \mathbb{E}[\zeta | X_0 = i] \le \frac{1}{1 - \|A\|_\infty}.$$

2 A simple stochastic integral

The notion of a *stochastic integral* arises in many contexts, ranging from atomic physics to engineering and finance. In this problem, we will consider the simplest such construction to illustrate the key ideas.

Let $W = (W_t)_{t\geq 0}$ be a Wiener process (a.k.a. Brownian motion) with parameter $\sigma^2 = 1$. Our goal is to give meaning to an integral of the form

$$I(f) = \int_0^T f(t)\mathrm{d}W_t$$

for any deterministic function $f : [0,T] \to \mathbb{R}$, which is square-integrable, i.e.,

$$\|f\|_{L^2[0,T]} \triangleq \left(\int_0^T |f(t)|^2 \mathrm{d}t\right)^{1/2} < \infty.$$

Let $S[0,T]$ be the space of all *step functions* on the interval $[0,T]$ – that is, $f \in S[0,T]$ if and only if there exist finitely many time instants $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$, such that $f(t)$ is constant on each interval $[t_i, t_{i+1})$, $i = 0, \ldots, n-1$. For such an f, we *define*

$$I(f) \triangleq \sum_{i=0}^{n-1} f(t_i)\left(W_{t_{i+1}} - W_{t_i}\right).$$

Observe that, while f was deterministic, $I(f)$ is a *random variable*. Now, one of the basic results in functional analysis is that the step functions in $\mathcal{S}[0,T]$ are *dense* in $L^2[0,T]$, i.e., for any $f \in L^2[0,T]$ we can find a sequence $\{f_n\}_{n=1}^{\infty}$ of step functions, such that $\|f_n - f\|_{L^2[0,T]} \xrightarrow{n\to\infty} 0$. Using this fact, we define the stochastic integral $I(f) = \lim_{n\to\infty} I(f_n)$. We will not worry about the details of this passage to the limit, but will instead investigate the properties of the integral $I(f)$ on $S[0,T]$.

(a) Prove the following properties:

(i) For any two $f_1, f_2 \in S[0,T]$ and any pair of coefficients c_1, c_2, $I(c_1 f_1 + c_2 f_2) = c_1 I(f_1) + c_2 I(f_2)$; moreover, the random variables $I(f)$, $f \in S[0,T]$, are jointly Gaussian.

(ii) For any two disjoint subintervals A, B of $[0,T]$, the random variables $I(\mathbf{1}_A)$ and $I(\mathbf{1}_B)$ are independent.

(iii) For any $f_1, f_2 \in S[0,T]$,

$$\mathbb{E}[I(f_1)I(f_2)] = \langle f_1, f_2\rangle_{L^2[0,T]} \triangleq \int_0^T f_1(t)f_2(t)\mathrm{d}t,$$

and in particular $\mathbb{E}[I^2(f)] = \|f\|^2_{L^2[0,T]}$.

(iv) For any $f \in S[0,T]$, $I(f) \sim N(0, \|f\|^2_{L^2[0,T]})$.

(b) As noted earlier, all of these results continue to hold true for every $f \in L^2[0,T]$. With that in mind, prove the following: Let $\{\varphi_i\}_{i=1}^{\infty}$ be a complete orthonormal system of functions in $L^2[0,T]$, i.e., $\langle \varphi_i, \varphi_j\rangle_{L^2[0,T]} = \delta_{ij}$ for all i, j, where δ_{ij} is the Kronecker symbol. Then the random variables $I(\varphi_1), I(\varphi_2), \ldots$ are i.i.d. $N(0,1)$.